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A CONSTRUCTION OF THE MEASURABLE POISSON BOUNDARY: FROM DISCRETE TO CONTINUOUS GROUPS

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Abstract. Let Γ be a dense countable subgroup of a locally compact continuous group G, and μ a probability measure on Γ. Two spaces of harmonic functions are naturally associated with μ : the space of μ-harmonic functions on the countable group Γ and the space of μ-harmonic functions seen as functions on G defined a.s. with respect to its Haar measure λ. Correspondingly we have two natural Poisson boundaries : the Γ-Poisson boundary and the G-Poisson boundary. Since boundaries on the countable group are quite well understood, a natural question is to ask how the G-boundary is related to the Γ-boundary.

In this paper we introduce a general technique that allows to build the G-Poisson boundary from the Γ-boundary. As an application, we determine the Poisson boundary of the closure of the Baumslag-Solitar group in the group of real matrices. In particular we show that, under suitable moment conditions and assuming that the action on $\mathbb{R}$ is not contracting, this boundary is the p-solenoid.

An important topics in the study of random walks on groups is the study of harmonic functions relative to a measure μ on a group G, i.e. of the functions f on the group such that

$$f(g) = \int_G f(g\gamma)d\mu(\gamma). \tag{1}$$

The Poisson Boundary is, in this setting, the measurable space that gives the integral representation of all bounded harmonic functions. This spaces encodes the asymptotic information contained in all random walk paths of law μ. A natural problem is to determine when this space is trivial and, if it is not, to exhibit a geometric model.

After the works of Blakwell, Choquet and Deny on abelian groups and the seminal papers of Furstenberg in the sixties, much progress has been made on these questions. In particular when the harmonic functions live on a *countable discrete group* Γ, a complete theory has been developed by Derriennic [5], Kaimanovich and Vershik [15], allowing to construct the Poisson Boundary (or at least decide whether it is trivial) for large classes of groups.

In the more general case where the measure μ is supported on a locally compact group G, the situation is more complex and one has to decide on which space harmonic functions live. A natural choice is to consider harmonic functions as a subspace of the space $L^\infty(G, \lambda)$ of essentially bounded functions with respect to the Haar measure λ of the group. If the measure μ is spread-out (and thus well adapted to the continuous structure) satisfactory general results have been obtained for Lie groups. The case where the measure μ is not necessarily smooth, is far from being completely understood. Some results have been obtained for particular classes of groups (e.g. Nilpotent groups [10, 2], NA groups [17]...). Abstract constructions have been also proposed, but they do not allow, in general, to construct geometric models for the boundary, nor to check whether it is trivial. I refer to the survey of M.Babillot [1] for a precise and complete overview of the subject and a more detailed bibliography.

In antithesis to the case of a smooth measure, we may consider a purely atomic measure μ, supported on a countable subgroup Γ that we can suppose dense in the continuous group G. In such situation, harmonic functions can be seen both as functions on the discrete group Γ and as measurable functions on the continuous group G.

When the Poisson boundary of the discrete group Γ is known (so that we can describe Γ-harmonic functions), several natural questions concerning G-measurable harmonic functions arise:

- Which Γ-harmonic functions can be extended to a G-harmonic function?
- How are the Γ-Poisson Boundary and the G-Poisson Boundary related?
- If we know how G acts on the Γ-Poisson Boundary, can we give conditions that imply that there are no non-trivial G-harmonic function?

The goal of this manuscript is to investigate these questions. We are in particularly interested in groups of matrices with rational entries, embedded as subgroups of groups of real matrices. In this case the Poisson boundaries of the countable subgroups are well understood [4], while there are still many open questions concerning the Poisson boundaries of the corresponding real groups (see section 1 for more detailed examples).

In section 2, we give a general construction of the G-Poisson boundary as a space of Γ-ergodic components in the product of G and the Γ-boundary (Proposition 1). We use this construction to determine the real boundary in the case of the Baumslag Solitar group $BS(1,p)$, embedded as a dense subgroup of

$$\left\{ \begin{bmatrix} p^m & b \\ 0 & 1 \end{bmatrix} | m \in \mathbb{Z}, b \in \mathbb{R} \right\} = \mathbb{R} \rtimes \mathbb{Z}.$$

In particular, if μ is dilating on $\mathbb{R}$ it is known that the $BS(1,p)$- Poisson boundary is the p-adic field $\mathbb{Q}_p$ (thus there is no "real" component in the boundary), however the real Poisson boundary is not trivial and is given by the p-solenoid

$$[0,1) \times \mathbb{Z}_p = (\mathbb{R} \times \mathbb{Q}_p) \Big/ \mathbb{Z}(1/p)$$

where the action of $\mathbb{Z}(1/p)$ on $\mathbb{R} \times \mathbb{Q}_p$ is the diagonal action (Corollary 3).

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1. G-harmonic functions and G-Poisson boundary

This section is a brief introduction to measurable Poisson boundary, following Babillot [1] and Kaimanovich [13]

G-harmonic functions. Let G be locally compact second countable (thus metrizable and complete) group. Let $\mathfrak{G}$ be the Borel σ-algebra of G and λ the right Haar measure.

Let μ be a probability on G such that the closed semigroup generated by the support of μ is the whole group G.

We say that a function $f \in L^\infty(G, \lambda)$ is *μ-harmonic on* (G, λ) (or *G-harmonic*) if

$$f(g) = \int_G f(g\gamma) d\mu(\gamma) \qquad \text{for } \lambda\text{-almost all } g \in G.$$

We denote by $H^\infty_\lambda(G)$ the subspace of G-harmonic functions in $L^\infty(G, \lambda)$.

It can be shown, using left convolution by identity approximations of G, that any $f \in H^\infty_\lambda(G)$ is λ-a.e. limit of harmonic functions that are left uniformly continuous on G. In this sense, the space of G-harmonic functions is determined by the behavior of continuous ones. In particular, if all continuous harmonic functions are constant then $H^\infty_\lambda(G)$ is trivial. We denote by $H^\infty_{\text{luc}}(\Gamma)$ the space of left uniformly continuous G-harmonic functions.

Random walks and invariant map. Harmonic functions can be seen as asymptotic values of random walks in the following way. Let $(\Omega, \mathbb{P}) = (G, \mu)^{\mathbb{N}}$ be the space of random steps and consider the right random walk

$$r_n(\omega) = \omega_1 \cdots \omega_n.$$

Let f be a bounded G-harmonic function. Notice that since the function f is defined only λ-almost surely, the process $f(gr_n(\omega))$ is well defined only for λ-almost all g. For this reason the starting point g has to be chosen according to ρ, a probability law on G with bounded density with respect to λ. Then the random process $f(gr_n(\omega))$ is well defined on the space $(G \times \Omega, \rho \times \mathbb{P})$, and since f is harmonic, it is a bounded martingale. Thus the limit

$$\lim_{n\to\infty} f(gr_n(\omega)) =: Z_f(g, \omega) \text{ exists } \rho(dg)\mathbb{P}(d\omega)\text{-almost surely.} \tag{2}$$

Let T be the shift on Ω; then is easily checked that

$$Z_f(g, \omega) = Z_f(g\omega_1, T\omega) \qquad \rho(dg)\mathbb{P}(d\omega)\text{-almost surely}$$

that is, Z_f is a bounded measurable *invariant map* on $G \times \Omega$. In fact (2) defines an isometry of $H^\infty_\lambda(G)$ onto the subspace of measurable invariant maps of $L^\infty(G \times \omega, \rho \times \mathbb{P})$. The reverse map is given by

$$f_Z(g) := \mathbb{E}(Z(g, \omega)) \qquad \rho(dg)\text{-almost surely.}$$

Poisson transform and G-Poisson boundary. Take a measurable space $(X, \mathfrak{X}, \nu)$ endowed with a measurable G-action and a μ-stationary probability measure ν. The *Poisson transform*

$$\mathcal{P}_\nu : \phi \mapsto f_\phi(g) := \int \phi(g \cdot x) d\nu(x)$$

maps any bounded function ϕ in $L^\infty(X, \rho * \nu)$ to a μ-harmonic function f_ϕ of $H^\infty_\lambda(\Gamma)$.
Notice that the Poisson transform is not well defined as a map on $L^\infty(X, \nu)$. In fact, since ν is not in general G-quasi invariant (i.e. $g * \nu$ is not in general absolutely continuous with respect to ν), two functions that coincide ν-a.s. can have different images.

If the Poisson transform is an **isometry** of $L^\infty(X, \rho * \nu)$ **onto** $H^\infty_\lambda(G)$ then we say that (X, ν) is the (G, μ)-*Poisson boundary*. It can be shown that the Poisson boundary is unique as a G-measurable space and that $(X, \rho * \nu)$ is a Lebesgue space (cf. [1] propositions 2.26 and 2.28).

If X is the G-Poisson boundary then there exists a measurable *boundary map* $\mathbf{bnd} : \Omega \to X$ such that for every harmonic function $f \in H^\infty_\lambda(G)$ there exists $\phi_f \in L^\infty(X, \rho * \nu)$ with

$$\phi_f(g \cdot \mathbf{bnd}(\omega)) = \lim_{n\to\infty} f(gr_n(\omega)) \qquad \rho(dg)\mathbb{P}(d\omega) - \text{a.s.}.$$

(cf. [1] proposition 2.26. See also the proof of Lemma 2).Thus

$$\begin{aligned} f(g) &= \int \phi_f(g \cdot x) d\nu(x) \qquad \rho(dg) - \text{a.s.} \qquad \text{and} \\ Z_f(g,\omega) &= \phi_f(g \cdot \mathbf{bnd}(\omega)) \qquad \rho(dg)\mathbb{P}(d\omega) - \text{a.s.}. \end{aligned}$$

The μ-invariant measure ν on X is then the image of $\mathbb{P}$ under **bnd**. The boundary map is G-equivariant in the sense that $\mathbf{bnd}(\omega) = \omega_1 \cdot \mathbf{bnd}(T\omega)$.

Countable group Γ. Suppose now that the group $G = \Gamma$ is countable. The Haar measure λ is then the counting measure and one can choose ρ with non-zero mass in all elements $g \in \Gamma$. This means that all the equalities above hold for all $g \in \Gamma$.

In this particular case (and under the hypothesis that the support of μ generates Γ as a semigroup) the stationary measure ν on X is Γ-quasi invariant and $\mathcal{P}_\nu$ is well defined on $L^\infty(X, \nu)$ itself.

The fact that μ is absolutely continuous with respect to λ_Γ is also fundamental for the study of Poisson boundary based on entropy [5, 15]. This complete theory has permitted to determine a geometric model of the Poisson boundary for large classes of countable groups.

Countable subgroup Γ of a continuous G. In this note we are interested in the case when the measure μ is supported on a countable subgroup Γ of a continuous group G and in particular when Γ is dense in G. Then a *continuous* harmonic function f on G is uniquely determined by the values $f(\gamma)$ for $\gamma \in \Gamma$. Thus f can also be seen as a Γ-harmonic function. In other words, the restriction to Γ is an isometric embedding of $H^\infty_{\text{luc}}(G)$ into $H^\infty_\lambda(\Gamma)$. In particular if (X, ν) is the Γ-Poisson boundary then there exists ϕ in $L^\infty(X, \nu)$ such that

$$f(\gamma) = \int_X \phi(\gamma \cdot x) \qquad \forall \gamma \in \Gamma.$$

However, in general there is no such integral representation for $f(g)$ when g is not in Γ, since X is not a priori a G-space.

In conclusion the Γ-Poisson boundary contains in principle all the information about the G-Poisson boundary. But in order to extract this information one needs to answer two related questions:

- Determine the G-action on (an extension of) X adapted to the action of G on $H^\infty_\lambda(G)$
- Determine which are the functions in $L^\infty(X, \nu)$ whose Poisson transform can be extended to G.

Examples: Linear groups with rational coefficients. We are in particular interested in the case where the Γ-Poisson boundary is known, but G-harmonic functions are not completely understood. Here some examples.

Affine groups. The real affine group $\mathrm{Aff}(\mathbb{R})$ is the group of real maps $(b,a) : x \mapsto ax+b$ with $a \in \mathbb{R}_+^*$ and $b \in \mathbb{R}$, that is the group of matrices

$$\mathrm{Aff}(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} | a \in \mathbb{R}_+^*, b \in \mathbb{R} \right\} = \mathbb{R} \rtimes \mathbb{R}_+^*.$$

Harmonic functions on $\mathrm{Aff}(\mathbb{R})$ have been widely studied and some results are known also without continuity assumptions on the measure μ. In particular, under the log-moment assumptions :

$$\mathbb{E}(|\log a|) < \infty \text{ and } \mathbb{E}(\log^+ b) < \infty,$$

it is known that:

- If $\mathbb{E}(\log a) = 0$ the $\mathrm{Aff}(\mathbb{R})$-Poisson Boundary is trivial ([18], see also [1, sect.4.5])
- If $\mathbb{E}(\log a) < 0$ the $\mathrm{Aff}(\mathbb{R})$-Poisson Boundary is $\mathbb{R}$ with the μ-invariant measure ν given by the law of

$$Z_\infty = \sum_{n=1}^{\infty} a_1 \cdots a_{n-1} b_n \tag{3}$$

where (b_n, a_n) are i.i.d. with law μ ([17], see also [1, thm 5.7]).

If $\mathbb{E}(\log a) > 0$ and the measure is spread-out then the $\mathrm{Aff}(\mathbb{R})$-Poisson Boundary is trivial. But it is still not known on what happens if $\mathbb{E}(\log a) > 0$ and the measure μ is supported on a countable subgroup Γ.

On the other hand, using entropic criteria, the Γ-Poisson boundaries are well understood. If $\Gamma = \mathrm{Aff}(\mathbb{Q})$, the group of affine maps with rational coefficients, and under suitable moment conditions, the $\mathrm{Aff}(\mathbb{Q})$-Poisson boundary is given by the product of the p-adic fields $\mathbb{Q}_p$ where the sum (3) converges a.s., that is

$$\prod_{p:\mathbb{E}(\log |a|_p)<0} \mathbb{Q}_p,$$

where we use the convention that $\mathbb{Q}_\infty = \mathbb{R}$ (see [3]).

This property was first proved by V.Kaimanovich [13] in the case of the Baumslag-Solitar group

$$BS(1,p) = \left\langle \begin{bmatrix} p^{\pm 1} & \pm 1 \\ 0 & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} p^m & qp^n \\ 0 & 1 \end{bmatrix} | m, n \text{ et } q \in \mathbb{Z} \right\} = \mathbb{Z}(\frac{1}{p}) \rtimes \mathbb{Z}.$$

for some prime p. In this particular case the $BS(1,p)$-Poisson boundary is $\mathbb{R}$ if $\mathbb{E}(\log a) = -\mathbb{E}(\log |a|_p) < 0$ and $\mathbb{Q}_p$ if $\mathbb{E}(\log |a|_p) = -\mathbb{E}(\log a) < 0$.

It is then natural to ask which harmonic functions can be extended to (continuous) harmonic functions of the closure of $BS(1,p)$ in $\mathrm{Aff}(\mathbb{R})$, that is to

$$\mathrm{Aff}(p,\mathbb{R}) = \left\{ \begin{bmatrix} p^m & b \\ 0 & 1 \end{bmatrix} | m \in \mathbb{Z}, b \in \mathbb{R} \right\} = \mathbb{R} \rtimes \mathbb{Z}.$$

It turns out that, even if the $BS(1,p)$-Poisson boundary is $\mathbb{Q}_p$, the real Poisson boundary is not trivial. In Corollary 3 we will construct the $\mathrm{Aff}(p,\mathbb{R})$-Poisson boundary as a p-solenoid.

The unpublished manuscript [16] of J.-F. Quint presents a similar example of dynamical system acting in non contacting way on the torus and constructs harmonic functions on the unstable variety.

As we will see in Corollary 2 this kind of construction is possible since the action of $BS(1,p)$ on $\mathrm{Aff}(p,\mathbb{R})\times\mathbb{Q}_p$ has a discrete orbit. It is still not clear to me what happens when the action of Γ on the product of G and the Γ-Poisson boundary is dense.

Question. For instance, let $\mathrm{Aff}(1/2,1/3)$ be the countable subgroup generated by the affinities

$$\left\langle \begin{bmatrix} 3^{\pm1} & \pm1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2^{\pm1} & \pm1 \\ 0 & 1 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 2^{m_2}3^{m_3} & q2^{n_2}3^{n_3} \\ 0 & 1 \end{bmatrix} | m_i, n_i \text{ et } q\in\mathbb{Z} \right\}.$$

Suppose $\mathbb{E}(\log|a|_\infty)>0$, thus the Γ-Poisson boundary is equal to $\mathbb{Q}_2$, $\mathbb{Q}_3$ or $\mathbb{Q}_2\times\mathbb{Q}_3$ (according to the sign of $\mathbb{E}(\log|a|_2)$ and $\mathbb{E}(\log|a|_3)$) and has no real component. Is then the $\mathrm{Aff}(\mathbb{R})$-Poisson boundary trivial?

Semi-simple groups. Similar questions arise for semi-simple groups. Take, for instance, a measure μ supported on $SL_2(\mathbb{Q})$. Then the $SL_2(\mathbb{Q})$-boundary is the product of the $\mathbb{Q}_p$-projective lines for all primes p such that the support of μ is not contained in a compact subgroup of $SL_2(\mathbb{Q}_p)$ (see [4]). In particular for

$$\Gamma = SL_2(\mathbb{Z}(1/2)) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | ad-cd=1, a,b,c \text{ et } d\in\mathbb{Z}/2^m \text{ for some } m\in\mathbb{Z} \right\}$$

the Γ-Poisson boundary is $\mathbb{P}^1(\mathbb{R})\times\mathbb{P}^1(\mathbb{Q}_2)$. It is natural to expect that the $SL_2(\mathbb{R})$-Poisson boundary should be $\mathbb{P}^1(\mathbb{R})$, however I am not aware of any proof of this fact. See also [1] section 1.7.4, for a similar example.

2. From Γ-boundaries to G-boundaries

Construction of a G-action on a Γ-space. Let $(X,\mathfrak{X},\nu)$ be a Γ-measurable Lebesgue space equipped with a measure ν that is Γ-quasi invariant. Suppose that Γ is contained in a locally compact group G. We want to construct a sort of minimal class of functions on X, on which G acts in such a way that the restriction to Γ of this action coincides with the Γ-action.

Consider the product space $(G\times X,\mathfrak{G}\times\mathfrak{X},\rho\times\nu)$ and define the Γ-action on $G\times X$

$$\gamma\star(g,x):=(g\gamma^{-1},\gamma\cdot x). \tag{4}$$

Let $\mathfrak{I}$ be the σ-algebra of $(\Gamma,\star)$-invariant functions of $G\times X$ that is the class of the functions ϕ such that $\rho(dg)\times\nu(dx)$-almost surely

$$\phi(g,x)=\phi(g\gamma^{-1},\gamma x) \qquad \forall\gamma\in\Gamma. \tag{5}$$

The σ-algebra $\mathfrak{I}$ is complete because $\rho\times\nu$ is $(\Gamma,\star)$-quasi invariant and Γ is countable. Rokhlin's correspondence associates with the σ-algebra $\mathfrak{I}$ a partition of $G\times X$, such that the functions in $\mathfrak{I}$ are constant on the elements of the partition. Since ρ is in the class of the Haar measure, we can choose the partition η to be G-equivariant, for the G-action on $G\times X$ given by left multiplication on the G component. In fact we have the following

Lemma 1. **1.** *There exists a countable family $\{\phi_n\}_{n\in\mathbb{N}}$ of bounded functions dense in $L^1(G\times X,\mathfrak{I},\rho\times\nu)$ such that for any $x\in X$ the function $\phi_n(\cdot,x)$ is continuous on G and such that (5) hold for all $(g,x)\in G\times X$.*

2. *Let η be the partition defined by the equivalence relation*

$$(g_1, x_1) \sim (g_2, x_2) \Leftrightarrow \phi_n(\gamma g_1, x_1) = \phi_n(\gamma g_2, x_2) \quad \forall n \in \mathbb{N} \text{ and } \gamma \in \Gamma. \tag{6}$$

Then η is a measurable partition (i.e. countably generated) and the associated complete σ-algebra coincides with $\mathfrak{I}$. In particular, for all $\phi \in \mathfrak{I}$ there exists $\widetilde{\phi}$ defined on $\widetilde{X} = G \times X/\eta$ such that $\phi = \widetilde{\phi} \circ \eta$ $\rho \times \nu$-a.s.

3. *If G acts on $G \times X$ by the left multiplication on the G component, then such a partition is G-equivariant, i.e.*

$$g_0 \cdot \eta(g, x) = \eta(g_0 g, x) \qquad \forall g_0, g \in G \text{ and } x \in X$$

Proof. **1.** Since $(G \times X, \mathfrak{I}, \rho \times \nu)$ is a Lebesgue space , there exists a countable family $\{\varphi_i\}$ of bounded functions dense in the L^1-norm. Set $\varphi_i(g, x) \equiv 0$ on the set of the (g, x) on which (5) does not hold, in order to obtain a family of functions $\star$-invariant everywhere.

Take an approximation of the identity on G, i.e. a sequence of non-negative continuous functions α_n whose supports shrink to the identity e and such that $\|\alpha_n\|_1^\lambda = 1$. Let

$$\varphi_i^n(g, x) = \int_G \alpha_n(h)\varphi_i(hg, x)d\lambda(h)$$

It easy checked that the φ_i^n are still $\star$-invariant. By classical results, for any $x \in X$, the functions $\varphi_i^n(\cdot, x)$ are continuous and converge to $\varphi_i(\cdot, x)$ in $L^1(G, \rho)$ when n goes to ∞. Then

$$\begin{aligned}\lim_{n\to\infty} \|\varphi_i^n - \varphi_i\|_1^{\rho\times\nu} &= \lim_{n\to\infty} \int_X \|\varphi_i^n(\cdot, x) - \varphi_i(\cdot, x)\|_1^\rho \, d\nu(x) \\ &= \int_X \lim_{n\to\infty} \|\varphi_i^n(\cdot, x) - \varphi_i(\cdot, x)\|_1^\rho \, d\nu(x) = 0\end{aligned}$$

since $\|\varphi_i^n(\cdot, x) - \varphi_i(\cdot, x)\|_1^\rho$ is bounded by $2\|\varphi_i\|_\infty^{\rho\times\nu}$.
Thus $\{\varphi_i^n\}_{i,n}$ is a countable family of G-continuous functions dense in $L^1(G \times X, \mathfrak{I}, \rho \times \nu)$.

2. Let $\{I_i\}$ be a countable family of intervals of $\mathbb{R}$ that separates the points and let

$$B(n, i) := \phi_n^{-1}(I_i) \subseteq G \times X.$$

Then the partition η defined in (6) is generated by $\{\gamma B(n, i) | \gamma \in \Gamma, n, i \in \mathbb{N}\}$; in fact

$$\eta(x, g) = \bigcap_{(g,x)\in\gamma B(n,i)} \gamma B(n, i) \bigcap_{(g,x)\notin\gamma B(n,i)} \gamma B(n, i)^c.$$

Since by step **1** the complete σ-algebra generated by the sets $B(n, i)$ is $\mathfrak{I}$, by Rohlin's correspondence, we can conclude that any function of $\mathfrak{I}$ is almost surely constant on the elements of the partition.

3. Finally to prove G-equivariance of η, we need to verify that for every $g_0, g \in G$ and $x \in X$, we have that $(g', x') \in \eta(g, x)$ i.e.

$$\phi_n(\gamma g', x') = \phi_n(\gamma g, x) \quad \forall n \in \mathbb{N} \text{ and } \gamma \in \Gamma$$

if and only if $g_0 \cdot (g', x') = (g_0 g', x') \in \eta(g_0 g, x)$, i.e.

$$\phi_n(\gamma_0 g_0 g', x') = \phi_n(\gamma g_0 g, x) \quad \forall n \in \mathbb{N} \text{ and } \gamma_0 \in \Gamma.$$

This follows from the fact that the functions ϕ_n are G-continuous and that Γ is dense in G, letting $\gamma \to \gamma_0 g_0$ (resp. $\gamma_0 \to \gamma g_0^{-1}$).

$\square$

If η is defined as in (6), let

$$\widetilde{X} = G \times X/\eta$$

be the space of the η components. The projection $\eta : G \times X \to \widetilde{X}$ defines a natural σ-algebra on $\widetilde{X}$

$$\widetilde{\mathfrak{I}} = \left\{ A \subset G \times X | \eta^{-1}(A) \in \mathfrak{G} \times \mathfrak{X} \right\}.$$

By the previous lemma the completion of $\eta^{-1}(\widetilde{\mathfrak{I}})$ is $\mathfrak{I}$.

We have just proved that $\widetilde{X}$ has a natural structure of a G-space :

$$g_0 \cdot \eta(g, x) = \eta(g_0 g, x) \tag{7}$$

Since the functions ϕ_n are $\star$-invariant everywhere, η allows to "transfer" the Γ-action from G to X, in the sense that

$$\eta(g\gamma, x) = \eta(g, \gamma \cdot x) \quad \forall (g, x) \in G \times X.$$

In particular, the action of Γ on X and on its projection on $\widetilde{X}$ are related by

$$\gamma \cdot \eta(e, x) = \eta(\gamma, x) = \eta(e, \gamma \cdot x).$$

Let $\widetilde{\nu} = \eta_*(\delta_e \times \nu)$ be the image on $\widetilde{X}$ of the measure $\delta_e \times \nu$ by η , that is

$$\widetilde{\nu}(\widetilde{\phi}) := \int_X \widetilde{\phi}(\eta(e, x)) d\nu(x). \tag{8}$$

By (7) we have then that the image by η of $\rho \times \nu$ is $\rho * \widetilde{\nu}$:

$$\begin{aligned} \eta_*(\rho \times \nu)(\widetilde{\phi}) &= \int_{G \times X} \widetilde{\phi}(\eta(g, x)) d\rho(g) d\nu(x) = \int_{G \times X} \widetilde{\phi}(g \cdot \eta(e, x)) d\rho(g) d\nu(x) \\ &= \rho * \widetilde{\nu}(\widetilde{\phi}) \end{aligned}$$

In conclusion

Corollary 1. *The projection η induces an isometry between $L^\infty(G \times X, \mathfrak{I}, \rho \times \nu)$ and $L^\infty(\widetilde{X}, \widetilde{\mathfrak{I}}, \rho * \widetilde{\nu})$.*

In the next section, using this measure theoretical construction, we will build the G-boundary on the Γ-boundary and prove that, if the $\star$-action has a fundamental domain, this fundamental domain is the G-boundary. However, it is not clear to me how to construct a geometric model of this measure space when the Γ-action is "dense".

An interesting case is, for instance, when G acts on X and this action coincides with the Γ-action. Then $L^\infty(X, \rho * \nu)$ embeds isometrically in $L^\infty(G \times X, \mathfrak{I}, \rho \times \nu)$. In fact if $\psi \in L^\infty(X, \rho * \nu)$ then

$$\phi_\psi(g, x) := \psi(g \cdot x)$$

is clearly $\star$-invariant and this embedding is an isometry since

$$\|\phi_\psi\|_\infty = \lim_{p \to \infty} \Big(\int \phi_\psi(g, x)^p \rho(dg) \nu(dx) \Big)^{1/p} = \lim_{p \to \infty} \Big(\int \psi(y)^p \rho * \nu(dy) \Big)^{1/p} = \|\psi\|_\infty.$$

Question. However it is not clear under which conditions this map is surjective, that is when $\widetilde{X}$ coincide with X.

For instance, as a toy model, take $G = (\mathbb{R}, +)$, $X = \mathbb{R}$ and $\Gamma = \mathbb{Q}$. For which measure ν does $\widetilde{X} = \mathbb{R}$? This is true by if ν is a.c. with respect to the Lebesgue measure, but what happen for other measures?

What happen if $G = SL_2(\mathbb{R})$ $X = \mathbb{P}^1(\mathbb{R})$ and $\Gamma = SL_2(\mathbb{Z}(1/2))$ (or $\Gamma = SL_2(\mathbb{Q})$)?

From Γ-boundaries to G-boundaries. Suppose that the measure ν on X is μ-stationary. For every bounded function ϕ in $L^\infty(G \times X, \rho \times \nu)$ define the Poisson transform:

$$\mathcal{P}_\nu : \phi \mapsto f_\phi(g) = \int \phi(g, x) d\nu(x) \text{for } \lambda(dg)\text{-almost all } g.$$

If $\phi \in \mathfrak{I}$ then f_ϕ is a bounded μ-harmonic function on $L^\infty(G, \lambda)$, as required. Indeed

$$\begin{aligned} f_\phi(g) &= \int \phi(g, x)\nu(dx) = \int \phi(g, \gamma \cdot x)\nu(dx)\mu(d\gamma) = \\ &= \int \phi(g\gamma, x)\nu(dx)\mu(d\gamma) = \int f_\phi(g\gamma)\mu(d\gamma). \end{aligned}$$

The following proposition shows that all G-harmonic functions can be written in such a way

Proposition 1. *If (X, ν) is the Poisson boundary of (Γ, μ) then for every μ-harmonic function f in G, there exists a bounded function $\phi \in \mathfrak{I}$ such that $f = f_\phi$ in $L^\infty(G, \lambda)$.*
In this case $\mathcal{P}_\nu$ is an isometry from $L^\infty(G \times X, \mathfrak{I}, \rho \times \nu)$ onto $H^\infty_\lambda(G)$. In other word $(\widetilde{X}, \widetilde{\nu})$ is the G-Poisson boundary.

Proof. Let $\omega \in (\Omega, \mathbb{P}) = (\Gamma^{\mathbb{N}}, \mu^{\otimes \mathbb{N}})$ and $r_k = r_k(\omega) = \omega_1 \cdots \omega_k$ be the right random walk on Γ of law μ. The process $f(gr_k(\omega))$ is a bounded martingale on the space $(G \times \Omega, \rho \times \mathbb{P})$ thus it converges almost surely. If $\mathbf{bnd} : \Omega \to X$ is the boundary map

$$\lim_{n \to \infty} f(gr_k(\omega)) = \phi(g, \mathbf{bnd}(\omega)) \tag{9}$$

$\rho \times \mathbb{P}$-almost surely. Thus $\phi(g, \mathbf{bnd}(\omega))$ is $G \times \Omega$ measurable and, since $\nu = \mathbf{bnd}^{-1}\mathbb{P}$, the function $\phi(g, x)$ is $G \times X$-measurable. Furthermore since Γ is countable, for $\rho \times \mathbb{P}$-almost all (g, ω)

$$\lim_{n \to \infty} f(g\gamma r_k(\omega)) = \phi(g\gamma, \mathbf{bnd}(\omega)) \quad \text{for all } \gamma \in \Gamma.$$

Since X is a μ-boundary, notice that

$$\omega_1 \mathbf{bnd}(T\omega) = \mathbf{bnd}(\omega)$$

where T is the shift on Ω. Take γ_1 in the support of μ then the event $\gamma_1 = \omega_1$ has positive measure and conditioned to this event

$$\phi(g\gamma_1^{-1}, \gamma_1 \mathbf{bnd}(T\omega)) = \phi(g\gamma_1^{-1}, \mathbf{bnd}(\omega)) = \lim_{n \to \infty} f(g\gamma_1^{-1}\gamma_1 r_n(T\omega)) = \phi(g, \mathbf{bnd}(T\omega))$$

Since $T\omega$ is independent of ω_1 and of same law as ω and that the support of μ generates Γ, we can conclude that $\phi \in \mathfrak{I}$.

Lastly, lets us check that the Poisson transform is an isometry. In fact

$$\|f_\phi\|_\infty^\rho = \lim_{p\to\infty}\left(\int |f_\phi(g)|^p d\rho(g)\right)^{1/p} \le \lim_{p\to\infty}\left(\int\int |\phi(g,x)|^p d\nu(x)d\rho(g)\right)^{1/p} = \|\phi\|_\infty^{\rho\times\nu}$$

On the other hand by the bounded convergence theorem

$$\begin{aligned}
\|\phi\|_p^{\rho\times\nu} &= \left(\int\int |\phi(g,x)|^p d\nu(x)d\rho(g)\right)^{1/p} = \\
&= \left(\int\int |\lim_{n\to\infty} f(gr_n(\omega))|^p d\mathbb{P}(\omega)d\rho(g)\right)^{1/p} = \\
&= \lim_{n\to\infty}\left(\int\int |f(gr_n(\omega))|^p d\rho(g)d\mathbb{P}(\omega)\right)^{1/p} \\
&\le \left(\int (\|f\|_\infty^\rho)^p d\mathbb{P}(\omega)\right)^{1/p} = \|f\|_\infty^\rho
\end{aligned}$$

since ρ is G-quasi invariant. □

G-Poisson boundary as Γ-ergodic diagonal components. Another way to express the result of Proposition 1 is to say that the G-Poisson boundary coincides with the space of ergodic components of Γ on $(G\times X)$ with respect to the action $\star$ defined in (4).

Observe that the action $\star$ is, in reality, the standard left diagonal action of Γ on $G\times X$:

$$\gamma \overset{\mathrm{d}}{\cdot} (g,x) = (\gamma g, \gamma\cdot x).$$

In fact the two actions are conjugated by the map $\pi : (g,x)\mapsto (g^{-1},x)$, that is an isomorphism of the measure space of $(G\times X, \rho\times\nu)$ that preserves the class of measure. Thus the space $L^\infty(G\times X,\mathfrak{I},\rho\times\nu)$ coincides (via π) with the space of bounded functions of $(G\times X,\rho\times\nu)$ that project on $\Gamma\backslash(G\times X)$. In particular the G-Poisson boundary is trivial if and only if the (diagonal) action of Γ on $(G\times X,\rho\times\nu)$ is ergodic.

Conversely if the action of Γ on $G\times X$ is "measurably discrete", that is there exists a fundamental domain Δ, then is possible to identify the G-Poisson boundary with this geometric model:

Corollary 2. *Suppose there exists a measurable fundamental domain $\Delta\in\mathfrak{G}\times\mathfrak{X}$ for the action $\star$ of Γ on $G\times X$ (or equivalently for the diagonal action) that is*

- $\rho\times\nu(\Gamma\star\Delta)=1$
- $\rho\times\nu(\Delta\cap\bigcup_{\gamma\in\Gamma-\{e\}}\gamma\star\Delta)=0$

Let $\mathfrak{D}$ be the restriction of the σ-algebra $\mathfrak{G}\times\mathfrak{X}$ to Δ. Then $L^\infty(\Delta,\mathfrak{D},\rho\times\nu)$ is isometric to $L^\infty(G\times X,\mathfrak{I},\rho\times\nu)$. The measurable space $(\Delta,\mathfrak{D})$ with the induced G-action

$$g_0 * \phi(g,x) := \sum_{\gamma\in\Gamma}\phi(g_0 g\gamma^{-1},\gamma\cdot x)1_\Delta(g_0 g\gamma^{-1},\gamma\cdot x) \text{ for all } \phi\in L^\infty(\Delta,\mathfrak{D},\rho\times\nu)$$

and the μ-invariant measure defined by

$$\widetilde{\nu}(\phi) := \sum_{\gamma\in\Gamma}\int \phi(\gamma^{-1},\gamma\cdot x)1_\Delta(\gamma^{-1},\gamma\cdot x)\nu(dx)$$

is the G-Poisson boundary.

Proof. The map
$$A \mapsto \Gamma \star A$$
induces an isometry of $L^\infty(\Delta, \mathfrak{D}, \rho \times \nu)$ onto $L^\infty(G \times X, \mathfrak{I}, \rho \times \nu)$.
In fact *if A is a non trivial set of Δ then $\Gamma \star A$ is a non trivial set of $\mathfrak{I}$.* Clearly $\Gamma \star A \in \mathfrak{I}$ and it has non-zero measure. Let $B \subset \Delta$ be a non-trivial set such that $\rho \times \nu(A \cap B) = 0$. We claim that $\rho \times \nu(\Gamma \star A \cap \Gamma \star B) = 0$; in fact the measure $\rho \times \nu$ being quasi-invariant of $\rho \times \nu(\gamma \star A \cap \gamma \star B) = 0$ and if $\gamma_1 \neq \gamma_2$
$$\rho \times \nu(\gamma_1 \star A \cap \gamma_2 \star B) \leq \rho \times \nu(\Delta \cap \bigcup_{\gamma \in \Gamma - \{e\}} \gamma \star \Delta) = 0$$
The isometry is surjective. Let $I \in \mathfrak{I}$, we claim that $I = \Gamma \star (I \cap \Delta)$. Indeed
$$\Gamma \star (I \cap \Delta) = \bigcup_\gamma \gamma \star I \cap \gamma \star \Delta = \bigcup_\gamma (I \cap \gamma \star \Delta) = I \cap \Gamma \star \Delta$$

Observe that if $A \subseteq \Delta$ then
$$1_{\Gamma \star A}(g, x) = \sum_{\gamma \in \Gamma} 1_A(g\gamma^{-1}, \gamma \cdot x)$$
and the sum has only one term for $\rho \times \nu$-almost all (g, x) . It easily seen that the projection of ν on $\mathfrak{D}$ is
$$\widetilde{\nu}(A) = \sum_{\gamma \in \Gamma} \int 1_A(\gamma^{-1}, \gamma \cdot x)\nu(dx) = \nu(\Gamma \star A).$$
□

3. G-Poisson boundary of Baumslag-Solitar group

Corollary 3. *Let p be a prime number and consider the Baumslag-Solitar group*
$$BS(1,p) = \left\langle \begin{bmatrix} p^{\pm 1} & \pm 1 \\ 0 & 1 \end{bmatrix} \right\rangle.$$
Let μ be a irreducible measure on $BS(1,p)$ with first logarithmic moment on $\mathbb{R}$ and $\mathbb{Q}_p$. Suppose that
$$\phi_p = \int_\Gamma \log |a(\gamma)|_p d\mu(\gamma) < 0$$
where $\gamma = \begin{bmatrix} a(\gamma) & b(\gamma) \\ 0 & 1 \end{bmatrix}$ that is the $BS(1,p)$-Poisson boundary is $X = \mathbb{Q}_p$. Let
$$\mathrm{Aff}(p, \mathbb{R}) = \left\{ \begin{bmatrix} p^m & b \\ 0 & 1 \end{bmatrix} | m \in \mathbb{Z}, b \in \mathbb{R} \right\} = \mathbb{R} \rtimes \mathbb{Z}$$
be the closure of $BS(1,p)$ in $\mathrm{Aff}(\mathbb{R})$. Then the $\mathrm{Aff}(p,\mathbb{R})$-Poisson boundary is the p-solenoid :
$$\Delta = \{(g, x) \in \mathrm{Aff}(\mathbb{R}) \times \mathbb{Q}_p | a(g) = 1; 0 \leq b(g) < 1; |x|_p \leq 1\} = [0, 1) \times \mathbb{Z}_p,$$
equipped with the $\mathrm{Aff}(p,\mathbb{R})$-action on $\phi \in L^\infty(\Delta, \rho \times \nu)$:
$$(b, p^m) \cdot \phi(x_\infty, x_p) = \sum_{\beta \in \mathbb{Z}(1/p)} 1_\Delta \cdot \phi(p^m x_\infty + b - \beta, p^m x_p + \beta),$$

and the invariant measure

$$\widetilde{\nu}(\phi) := \sum_{\beta\in\mathbb{Z}(1/p)\cap[0,1)} \int \phi(\beta, x-\beta)1_{\mathbb{Z}_p+\beta}(x)\nu(dx).$$

Proof. We just need to prove that Δ is a fundamental domain. In fact for any $x \in \mathbb{Q}_p$, let $\alpha(x) \in \mathbb{Z}(1/p)$ such that $|x-\alpha(x)|_p \leq 1$. The choice of α is unique up to the sum with an integer. It easily checked that, for every $(b,x) \in \mathbb{R}\times\mathbb{Q}_p$, the unique $k \in \mathbb{Z}(1/p)$ such $|x+k|_p \leq 1$ and $b-k \in [0,1)$ is $k = [b+\alpha(x)] - \alpha(x)$. Thus

$$\gamma \star ((b,p^m),x) \in \Delta \Leftrightarrow \gamma = ([b+\alpha(p^m x)] - \alpha(p^m x), p^m)$$

□

To illustrate how the previous corollary can be used to study the behaviour of harmonic functions on $BS(1,p)$, consider for example

$$\phi(g,x) = 1_{[0,1)\times\{1\}}(g)1_{p\mathbb{Z}_p}(x)$$

and the associated harmonic function:

$$f(b,p^m) = \int \sum_{\beta\in\mathbb{Z}(1/p)} 1_{[0,1)}(b-\beta)1_{p\mathbb{Z}_p}(p^m x+\beta)\nu(dx).$$

Then we have

- f is periodic of period p on the b coordinate

$$f(pk+b,p^m) = \int \sum_{\beta\in\mathbb{Z}(1/p)} 1_{[0,1)}(b-\beta)1_{p\mathbb{Z}_p}(p^m x+\beta+pk)\nu(dx) = f(b,p^m)$$

- $\lim_{m\to+\infty} f(b,p^m) = 1$ if $b \in [0,1)+p\mathbb{Z}$. In fact $\|f\|_\infty = 1$ and $b \in [0,1)$

$$f(b,p^m) \geq \int 1_{[0,1)}(b)1_{p\mathbb{Z}_p}(p^m x)\nu(dx) = \nu(p^{1-m}\mathbb{Z}_p) \to 1$$

  when $m \to +\infty$
- $\lim_{m\to+\infty} f(b,p^m) = 0$ if $b \notin [0,1)+p\mathbb{Z}$ in fact

$$\begin{aligned} f(b,p^m) &\leq \int \sum_{\beta\in\mathbb{Z}(1/p)} 1_{[0,1)}(b-\beta)1_{p\mathbb{Z}_p}(p^m x+\beta)1_{p^{1-m}\mathbb{Z}_p}(x)\nu(dx) + \\ &\qquad +(1-\nu(p^{1-m}\mathbb{Z}_p)) \\ &\leq \sum_{\beta\in\mathbb{Z}(1/p)} 1_{[0,1)}(b-\beta)1_{p\mathbb{Z}_p}(\beta) + (1-\nu(p^{1-m}\mathbb{Z}_p)) \\ &= \sum_{k\in\mathbb{Z}} 1_{[0,1)}(b-pk) + (1-\nu(p^{1-m}\mathbb{Z}_p)) \\ &= 1_{[0,1)+p\mathbb{Z}}(b) + (1-\nu(p^{1-m}\mathbb{Z}_p)). \end{aligned}$$

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