

Brownian Motion and Harmonic Functions on $\text{Sol}(p, q)$ [†]

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The Lie group $\text{Sol}(p, q)$ is the semidirect product induced by the action of \mathbb{R} on \mathbb{R}^2 which is given by $(x, y) \mapsto (e^{pz}x, e^{-qz}y)$, $z \in \mathbb{R}$. Viewing $\text{Sol}(p, q)$ as a three-dimensional manifold, it carries a natural Riemannian metric and Laplace–Beltrami operator. We add a linear drift term in the z -variable to the latter, and study the associated Brownian motion with drift. We derive a central limit theorem and compute the rate of escape. Also, we introduce the natural geometric compactification of $\text{Sol}(p, q)$ and explain how Brownian motion converges almost surely to the boundary in the resulting topology. We also study all positive harmonic functions for the Laplacian with drift, and determine explicitly all minimal harmonic functions. All these are carried out with a strong emphasis on understanding and using the geometric features of $\text{Sol}(p, q)$, and, in particular, the fact that it can be described as the horocyclic product of two hyperbolic planes with curvatures $-p^2$ and $-q^2$, respectively.

1 Introduction

$\text{Sol}(p, q)$ is the group of all matrices of the form

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[†]Dedicated to Alessandro Figà-Talamanca on the occasion of his 70th birthday.

$$\mathfrak{g} = \begin{pmatrix} e^{pz} & x & 0 \\ 0 & 1 & 0 \\ 0 & y & e^{-qz} \end{pmatrix}, \quad x, y, z \in \mathbb{R}. \quad (1.1)$$

The parameters p and q are positive real numbers. It will be useful to think separately of $\text{Sol}(p, q)$ as a Lie group and as a manifold. In the latter situation, we shall often write $\mathfrak{g} = (x, y, z)$ or also \mathfrak{r} or \mathfrak{h} for its elements, instead of \mathfrak{g} . Its natural length element is

$$ds^2 = d_{p,q}s^2 = e^{-2pz} dx^2 + e^{2qz} dy^2 + dz^2, \quad (1.2)$$

which is invariant under the left action of $\text{Sol}(p, q)$ on itself as an isometry group. If we identify the element \mathfrak{g} of (1.1) with (x, y, z) , then $\text{Sol}(p, q)$ is \mathbb{R}^3 topologically (but of course not metrically). In those coordinates, the group product is

$$(a, b, c) \cdot (x, y, z) = (e^{pc}x + a, e^{-qc}y + b, c + z). \quad (1.3)$$

The manifold/Lie group $\text{Sol}(1, 1)$ with the above metric has been attracting attention as one of the eight model geometries of Thurston's geometrization (ex-)conjecture, see Thurston [45, 46] and Scott [42].

The purpose of this case study is to describe the behavior of Brownian motion in space and time, and to determine all positive harmonic functions on $\text{Sol}(p, q)$ with respect to its Laplace–Beltrami operator and the variant where a “vertical” drift term (in z) is added to the latter. More precisely, we shall derive a central limit theorem for Brownian motion with drift, describe convergence of this process to the natural geometric boundary at infinity, and we shall determine all positive eigenfunctions of those Laplacians. The experienced reader will know how intimately such stochastic and potential theoretic features are linked with each other.

Before we can explain the results, we need some details. To start, let $\mathbb{H} = \{x + iw : x \in \mathbb{R}, w > 0\}$ be hyperbolic upper half plane with the standard length element $w^{-2}(dx^2 + dw^2)$. We can pass to the logarithmic model by substituting $z = \log w$, and in those coordinates the length element becomes $e^{-2z} dx^2 + dz^2$. Now we also change curvature by modifying the length element into

$$ds^2 = d_p s^2 = e^{-2pz} dx^2 + dz^2.$$

We write $\mathbb{H}(p)$ for the hyperbolic plane with this parametrization and metric. Then we have the natural projections

$$\begin{aligned}\pi_1 : \text{Sol}(p, q) &\rightarrow \mathbb{H}(p), & (x, y, z) &\mapsto (x, z), \\ \pi_2 : \text{Sol}(p, q) &\rightarrow \mathbb{H}(q), & (x, y, z) &\mapsto (y, -z).\end{aligned}\tag{1.4}$$

The *horocycle at level z* in $\mathbb{H}(p)$ is the set $\{(x, z) : x \in \mathbb{R}\}$, and we write $\tilde{\pi}(x, z) = z$. Thus, we obtain another natural projection $\tilde{\pi} : \mathbb{H}(p) \rightarrow \mathbb{R}$. We also consider $\tilde{\pi}$ as a projection of $\text{Sol}(p, q)$ onto \mathbb{R} , where $\tilde{\pi}(x, y, z) = z$. We shall write d for each of the metrics induced by the respective length elements; it will usually be evident from the context to which of the underlying spaces this refers—or else, that space will appear in the index. (On \mathbb{R} we then have $d_{\mathbb{R}}(z_1, z_2) = |z_1 - z_2|$.) Note that our projections preserve distances in the following sense:

$$\begin{aligned}d_{\text{Sol}}((x, y_1, z_1), (x, y_2, z_2)) &= d_{\mathbb{H}(q)}((y_1, -z_1), (y_2, -z_2)), \\ d_{\text{Sol}}((x_1, y, z_1), (x_2, y, z_2)) &= d_{\mathbb{H}(p)}((x_1, z_1), (x_2, z_2)), \quad \text{and} \\ d_{\text{Sol}}((x, y, z_1), (x, y, z_2)) &= |z_1 - z_2|.\end{aligned}\tag{1.5}$$

A main structural feature is that the manifold $\text{Sol}(p, q)$ comprises two hyperbolic planes (with respective curvatures $-p^2$ and $-q^2$) that are glued together by identifying opposite horocycles: it can be seen as the *horocyclic product* of $\mathbb{H}(p)$ and $\mathbb{H}(q)$,

$$\text{Sol}(p, q) = \{(u, v) \in \mathbb{H}(p) \times \mathbb{H}(q) : \tilde{\pi}(u) + \tilde{\pi}(v) = 0\},\tag{1.6}$$

with its metric arising naturally from those two hyperbolic planes.

We remark here that there are various different types of horocyclic products. $\text{Sol}(p, q)$ has two sister structures. One is the *Diestel–Leader graph* $\text{DL}(p, q)$, which is the horocyclic product of two regular trees with degrees $p + 1$ and $q + 1$, respectively, where $p, q \geq 2$ are integer. One of its interesting features is that when $p = q$, it is a Cayley graph of the *lamplighter group* $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$. Random walks and harmonic functions on $\text{DL}(p, q)$ have been studied intensively by Bertacchi [9], Woess [48], Bartholdi and Woess [6], and Brofferio and Woess [11, 12]. The other sister structure is *treebolic space* $\text{HT}(p, q)$, which is the horocyclic product of $\mathbb{H}(p)$ and the tree with degree $q + 1$, where $p > 0$ (real) and $q \geq 2$ (integer). When $p = q$, the Baumslag–Solitar group $\langle a, b \mid ab = b^a a \rangle$ acts on $\text{HT}(q, q)$ with compact quotient. The study of potential theory and Brownian motion on treebolic

space is harder than on Sol and on DL (where random walk replaces Brownian motion), first of all because of the conceptual and technical difficulty in constructing the right Laplacian(s) on the two-dimensional complex HT. This is an ongoing work of Bendikov et al. [7, 8].

Brownian motion and random walks on Sol(1, 1) made a brief appearance in the work of Lyons and Sullivan [36]. Harmonic functions for random walks on Sol(1, 1) also appear in Raugi [40, Exemple 2, p. 677].

HT, DL, and Sol are also objects of great interest in relation with geometric group theory. Quasi-isometries of those spaces have been studied by Farb and Mosher [21] (for HT(p, p)) and by Eskin et al. [19, 20] (for DL and Sol). The last two papers also contain a good description of several aspects of the geometry of Sol. The instructive article of Troyanov [47] contains a detailed study of various geometric features of Sol(1, 1), including curvature (which is positive in some directions and negative in others), the isometry group and in particular, the geodesic rays and their asymptotic classes. See also Kim [30], who basically provides a subset of the material of [47].

The Laplace operator with vertical drift parameter $\mathbf{a} \in \mathbb{R}$ on Sol(p, q) is

$$\mathfrak{L}_{\mathbf{a}} = \mathfrak{L}_{\mathbf{a}}^{\text{Sol}(p,q)} = \frac{1}{2} \left(e^{2pz} \frac{\partial^2}{\partial x^2} + e^{-2qz} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + \mathbf{a} \frac{\partial}{\partial z}. \quad (1.7)$$

The Laplace–Beltrami operator arises for $\mathbf{a} = (q - p)/2$.

As a matter of fact, this involves a small abuse of terminology: in differential geometry, the “true” Laplace–Beltrami operator would be *twice* the one which we are using. Here, we are following the probabilistic habits: with the factor $\frac{1}{2}$, in the standard Euclidean situation, the Laplacian is the infinitesimal generator of standard Brownian motion. The situation is similar here.

Under the projection π_1 , the operator $\mathfrak{L}_{\mathbf{a}}$ projects onto the operator on $\mathbb{H}(p)$ given by

$$\mathfrak{L}_{\mathbf{a}}^{\mathbb{H}(p)} = \frac{1}{2} \left(e^{2pz} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) + \mathbf{a} \frac{\partial}{\partial z}. \quad (1.8)$$

By “projects” we mean that for a C^2 -function f_1 on $\mathbb{H}(p)$, one (obviously) has $\mathfrak{L}_{\mathbf{a}}(f_1 \circ \pi_1) = (\mathfrak{L}_{\mathbf{a}}^{\mathbb{H}(p)} f_1) \circ \pi_1$. This is the Laplace–Beltrami operator on $\mathbb{H}(p)$ when $\mathbf{a} = -p/2$.

Analogously, under the projection π_2 (where the sign of z is changed), $\mathfrak{L}_{\mathbf{a}}$ projects onto the operator on $\mathbb{H}(q)$ given by

$$\mathfrak{L}_{-\mathbf{a}}^{\mathbb{H}(q)} = \frac{1}{2} \left(e^{2qz} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - \mathbf{a} \frac{\partial}{\partial z}. \quad (1.9)$$

And finally, \mathfrak{L}_a projects under $\tilde{\pi}$ onto the operator on \mathbb{R} given by

$$\tilde{\mathfrak{L}}_a = \frac{1}{2} \frac{d^2}{dz^2} + a \frac{d}{dz}. \quad (1.10)$$

Coming back to the outline of the contents of this paper, some basic preliminaries are laid out in Section 2. Our first, probabilistic object of study is then drifted Brownian motion $\mathfrak{Z}_t = (X_t, Y_t, Z_t)_{t>0}$ on $\text{Sol}(p, q)$, that is, the diffusion process whose infinitesimal generator is \mathfrak{L}_a . The projections of \mathfrak{Z}_t on $\mathbb{H}(p)$, $\mathbb{H}(q)$ and \mathbb{R} are the diffusions whose infinitesimal generators are the respective projected operators defined above.

In Section 3, we describe this process in terms of stochastic integrals and first derive a central limit theorem for (X_t, Y_t, Z_t) . Combining this with estimates from Section 2 for the metric of Sol , we also obtain a central limit theorem for $d(\mathfrak{Z}_t, o)$. Its form for the case $a = 0$ is somewhat different from what happens for $a \neq 0$. As a corollary, we obtain the linear rate of escape:

$$\frac{d(\mathfrak{Z}_t, o)}{t} \rightarrow |a| \quad \text{almost surely, as } t \rightarrow \infty,$$

where $o = (0, 0, 0)$. This is the same as the rate of escape for the projected (“vertical”) Brownian motion with drift $(Z_t)_{t>0}$ on \mathbb{R} , so that the lateral motion in the x - and y -variables does not contribute to that rate.

Since the Sol -group has exponential growth, our process is always transient, that is, with probability 1 it eventually leaves each compact set. Section 4 adds more details to the description of how our process tends to infinity in space. Namely, $\text{Sol}(p, q)$ has a natural geometric compactification: since $\text{Sol}(p, q)$ is a subset of the product of two hyperbolic planes (or equivalently, hyperbolic disks), it embeds naturally into the product of two closed unit disks, and the closure of $\text{Sol}(p, q)$ in this bi-disk is the compactification. Topologically, the resulting boundary at infinity has the shape of a filled number “8”, that is, two full closed disks glued together at a single (glueing) point. It is not a “visibility” boundary: neither the glueing point nor any of the interior points of the two disks come up as a limit of some geodesic ray in Sol ; all the other boundary points *are* limits of geodesic rays.

It is a rather straightforward, but nevertheless informative task to verify that Brownian motion tends almost surely in the topology of that compactification to a limit random variable that lives on the boundary at infinity of $\text{Sol}(p, q)$. If $a = 0$, then this

limit is the glueing point deterministically. Otherwise, that limit random variable lies on one of the two circles that make up the “8” (not their interiors) and its distribution is continuous. Thus, when $a \neq 0$, we (almost surely) have the geodesic ray from the origin to the random limit point. If $\gamma = (\gamma(t))_{t \geq 0}$ is that limit geodesic, then we show that for $a \neq 0$, the deviation of Z_t from that ray is at most logarithmic, that is, there is $c > 0$ such that

$$\limsup_{t \rightarrow \infty} d(Z_t, \gamma) / \log t \leq c \quad \text{almost surely.}$$

This result comprises the analogous one for Brownian motion with drift on the hyperbolic plane. The latter is a special case of a theorem stated by Ancona [1, Théorème 7.3]. There is a corresponding theorem for random walks on free groups, resp. trees, that was shown by Ledrappier [34]; see Woess [49, Theorem 9.59] for a general and simple proof.

The second main body of this work concerns positive harmonic functions. These are the positive C^2 -functions that are annihilated by the respective Laplacian. We can also handle positive eigenfunctions.

We start in Section 5 by displaying some of the potential theoretic, respectively, analytic ingredients that are needed. Then we prove in Section 6 that every positive \mathcal{L}_a -eigenfunction on $\text{Sol}(p, q)$ has the form

$$h(x, y, z) = h_1(x, z) + h_2(y, -z),$$

where h_1 is a nonnegative $\mathcal{L}_a^{\mathbb{H}(p)}$ -eigenfunction on $\mathbb{H}(p)$ and h_2 is a nonnegative $\mathcal{L}_{-a}^{\mathbb{H}(q)}$ -eigenfunction on $\mathbb{H}(q)$, both with the same eigenvalue as h .

This decomposition is not unique, but we can also see where nonuniqueness comes from, namely, harmonic functions that only depend on the “height” z . What we do is indeed to describe all minimal positive eigenfunctions, based on ideas from the discrete setting of Diestel–Leader graphs, see [12].

Since the positive $\mathcal{L}_a^{\mathbb{H}(p)}$ -eigenfunctions are known explicitly as integrals of modified Poisson kernels, the above result leads to a complete description of all positive \mathcal{L}_a -eigenfunctions on $\text{Sol}(p, q)$. Thus, the positive eigenfunctions of the Laplacian on $\text{Sol}(p, q)$ can be described fully in terms of modified Poisson kernels on each of the two hyperbolic planes that make up our space.

The computations undertaken here are related with the study of Martin compactifications of symmetric spaces, although the manifold $\text{Sol}(p, q)$ has both positive and negative sectional curvatures. The reader is referred to the book by Guivarc’h, Ji, and Taylor [24] and the survey by Kaimanovich [26] and the references therein. In particular,

we point out the work of Karpelevich [29] and Taylor [43]. For more general homogenous manifolds, see, for example, Babillot [2].

We get close to answering the question of Lyons and Sullivan [36] to determine the Martin boundary of Sol ; we find the minimal boundary and have a clear idea what the Martin compactification has to be.

Besides the work mentioned already, there is a rich literature on issues close to the ones studied here regarding Lie groups that are semidirect products of \mathbb{R} (or other Abelian groups) and Abelian or nilpotent groups. See Molchanov [38], Malliavin and Malliavin-Brameret [37], Damek and Hulanicki [14], or Damek et al. [15], to name just a few.

We want to underline that the main spirit of this paper is to study the outlined issues via strong use of the geometry of $\text{Sol}(p, q)$ in terms of the two hyperbolic planes and their horocyclic product.

2 Basic Facts

The first part of this section contains some basic facts regarding $\text{Sol}(p, q)$ that are quite straightforward. They are included here for the sake of the completeness of the picture; most proofs are omitted. We refer to [47] for a careful study of geometry and group properties of $\text{Sol}(1, 1)$, most of which carries over to $\text{Sol}(p, q)$.

Lemma 2.1. A natural base of the Lie algebra of the Lie group $\text{Sol}(p, q)$ is given by the vector fields

$$X = e^{pz}\partial_x, \quad Y = e^{-qz}\partial_y, \quad Z = \partial_z.$$

The Lie brackets are

$$[X, Y] = 0, \quad [Z, X] = pX, \quad [Z, Y] = -qY.$$

With X, Y , and Z as orthonormal frame, the associated left invariant metric is the one of (1.2). \square

Lemma 2.2. The sectional curvatures of the manifold $\text{Sol}(p, q)$ in terms of the vector fields X, Y, Z of Lemma 2.1 are given by

$$K(X, Z) = -p^2, \quad K(Y, Z) = -q^2, \quad K(X, Y) = pq. \quad \square$$

The computations are precisely as done in [47] in the case $p = q = 1$.

Lemma 2.3. The Riemannian volume element of the Sol-manifold is

$$d_3 = e^{(q-p)z} dx dy dz.$$

This is also the left Haar measure of $\text{Sol}(p, q)$ as a group. The modular function on this group is $\Delta(\mathfrak{g}) = e^{(q-p)\tilde{\pi}(\mathfrak{g})}$, where \mathfrak{g} is parametrized by (x, y, z) as in (1.1) and $\tilde{\pi}(\mathfrak{g}) = z$. The group is unimodular if and only if $p = q$. \square

Next, consider the group $\text{Aff}(p)$ of all matrices of the form

$$\begin{pmatrix} e^{pz} & x \\ 0 & 1 \end{pmatrix}, \quad x, z \in \mathbb{R}. \quad (2.1)$$

This is nothing but the group of orientation preserving *affine transformations* of hyperbolic plane, again parametrized by the logarithmic model and substituting the habitual upper left term e^z with e^{pz} . We can identify the group $\text{Aff}(p)$ with the surface $\mathbb{H}(p)$ in the same way as we identified Sol as a group with Sol as a manifold. By left multiplication, $\text{Aff}(p)$ acts isometrically on $\mathbb{H}(p)$. We recall the following.

Lemma 2.4. (a) The Riemannian area element of $\mathbb{H}(p)$ is $e^{-pz} dx dz$. This is also the left Haar measure on the group $\text{Aff}(p)$, and the modular function on $\text{Aff}(p)$ is $\Delta(\mathfrak{g}) = e^{p\tilde{\pi}(\mathfrak{g})}$, where $\mathfrak{g} = \begin{pmatrix} e^{pz} & x \\ 0 & 1 \end{pmatrix}$ and $\tilde{\pi}(\mathfrak{g}) = z$. \square

Let $D(2)$ be the group of reflections $(x, y, z) \mapsto (\pm x, \pm y, z)$. They all are isometries of the manifold $\text{Sol}(p, q)$. When $p = q$, also the reflection $(x, y, z) \mapsto (y, -x, -z)$ is an isometry (it is not when $p \neq q$), and we write $D(4)$ for the (dihedral) group generated by D and this mapping.

Lemma 2.5. When $p \neq q$, the full isometry group of the manifold $\text{Sol}(p, q)$ is the semidirect product of $D(2)$ with the group $\text{Sol}(p, q)$.

When $p = q$, the full isometry group is the semidirect product of $D(4)$ with the group $\text{Sol}(p, q)$. \square

For the case $p = q = 1$, see [47].

We can interpret the projections π_1 and π_2 of (1.4) as homomorphisms from the group $\text{Sol}(p, q)$ onto $\text{Aff}(p)$ and $\text{Aff}(q)$, respectively. In the same way, $\tilde{\pi}$ is a homomorphism onto the additive group \mathbb{R} .

Lemma 2.6. (a) The Laplacian \mathcal{L}_a on Sol(p, q) is reversible (self-adjoint) with respect to the measure

$$\mathbf{m}_a(d_{\mathfrak{z}}) = e^{(2a+p-q)z} d_{\mathfrak{z}} = e^{2az} dx dy dz.$$

(b) The Laplacian $\mathcal{L}_a^{\mathbb{H}(p)}$ on $\mathbb{H}(p)$ is reversible with respect to the measure

$$e^{2az} dx dz.$$

(c) The Laplacian $\tilde{\mathcal{L}}_a$ on \mathbb{R} is reversible with respect to the measure

$$e^{2az} dz. \quad \square$$

Our Laplacian is invariant under the group action of Sol. Let $g_0 = (a, b, c)$ be a group element, and define the translate of a function f on Sol as $\tau_{g_0} f(g) = f(g_0 g)$, that is,

$$\tau_{g_0} f(x, y, z) = f(e^{pc}x + a, e^{-qc}y + b, c + z). \quad (2.2)$$

Lemma 2.7. For any $g_0 \in \text{Sol}(p, q)$,

$$\mathcal{L}_a(\tau_{g_0} f) = \tau_{g_0}(\mathcal{L}_a f). \quad \square$$

The proof is completely elementary, using (2.2).

We shall need the following observations on the metric. Regarding our hyperbolic planes in the logarithmic model, let us remark here that the metric of $\mathbb{H}(p)$ is linked with the standard one of $\mathbb{H} = \mathbb{H}(1)$ by the formula

$$d_{\mathbb{H}(p)}((x, z), (x', z')) = \frac{1}{p} d_{\mathbb{H}(1)}((px, pz), (px', pz')). \quad (2.3)$$

While for Diestel–Leader graphs, there is an explicit formula for the graph metric in terms of the two underlying trees [9], we do not have such a formula on Sol. However, we have at least the following distance estimates.

Proposition 2.8. For all $\mathfrak{z} = (x, y, z) \in \text{Sol}(p, q)$, with $x, y \neq 0$ in (ii) and (iv),

(i)

$$d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z}) \geq |z|,$$

(ii)

$$d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z}) \geq 2 \frac{\log |x|}{p} + 2 \frac{\log |y|}{q} - |z| - \left(\frac{2}{p} + \frac{2}{q} \right) \log d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z}),$$

(iii)

$$\begin{aligned} d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z}) &\leq d_{\mathbb{H}(p)}((x, z), (0, 0)) + d_{\mathbb{H}(q)}((y, -z), (0, 0)) - |z| \\ &\leq c + \frac{2 \log(1 + |x|)}{p} + \frac{2 \log(1 + |y|)}{q} + |z| \end{aligned}$$

(iv)

$$\begin{aligned} d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z}) &\leq c' + \left| \frac{\log |x|}{p} + \frac{\log |y|}{q} \right| \\ &\quad + \min \left\{ \left| \frac{\log |x|}{p} \right| + \left| \frac{\log |y|}{q} + z \right|, \left| \frac{\log |x|}{p} - z \right| + \left| \frac{\log |y|}{q} \right| \right\}, \end{aligned}$$

where $c, c' > 0$. □

Proof. Inequality (i) is clear.

For (ii), Let $\mathfrak{z}(t) = (x(t), y(t), z(t))_{t \in [0, d]}$ be a geodesic path in Sol from \mathfrak{o} to \mathfrak{z} , where $d = d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z})$. Let

$$M = \max\{z(t) : t \in [0, d]\} \quad \text{and} \quad m = \min\{z(t) : t \in [0, d]\},$$

so that $M \geq 0$ and $m \leq 0$. Then

$$\begin{aligned} d_{\text{Sol}}(\mathfrak{z}, \mathfrak{o}) &= \int_0^d \sqrt{e^{-2pz(t)} \dot{x}(t)^2 + e^{2qz(t)} \dot{y}(t)^2 + \dot{z}(t)^2} dt \\ &\geq e^{-pM} \int_0^d \sqrt{\dot{x}(t)^2 + \dot{z}(t)^2} dt \geq e^{-pM} \sqrt{x^2 + z^2}. \end{aligned}$$

Thus,

$$pM \geq \log \sqrt{x^2 + z^2} - \log d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z}), \quad \text{and analogously} \quad -qm \geq \log \sqrt{y^2 + z^2} - \log d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z}).$$

Now let \mathfrak{z}_M and \mathfrak{z}_m be points on the geodesic from \mathfrak{o} to \mathfrak{z} with heights M and m , respectively. Then, according to which of the two “comes first”, (i) yields that either

$$d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z}) = d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z}_M) + d_{\text{Sol}}(\mathfrak{z}_M, \mathfrak{z}_m) + d_{\text{Sol}}(\mathfrak{z}_m, \mathfrak{z}) \geq M + (M - m) + (z - m), \quad \text{or}$$

$$d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z}) = d_{\text{Sol}}(\mathfrak{o}, \mathfrak{z}_m) + d_{\text{Sol}}(\mathfrak{z}_m, \mathfrak{z}_M) + d_{\text{Sol}}(\mathfrak{z}_M, \mathfrak{z}) \geq -m + (M - m) + (M - z).$$

We see that

$$d_{\text{Sol}}(o, \mathfrak{z}) \geq 2(M - m) - |z|,$$

and combining this with the above, we obtain (ii).

For proving the first part of (iii), we may suppose without loss of generality that $z \geq 0$. (Otherwise we exchange the roles of the two hyperbolic planes $\mathbb{H}(p)$ and $\mathbb{H}(q)$.) Also, since the reflections $(x, y, z) \mapsto (\pm x, \pm y, z)$ are isometries, we may assume that $x, y \geq 0$.

Note that in the logarithmic model of $\mathbb{H}(p)$, any geodesic arc is either vertical (i.e., of the form $t \mapsto (x_0, t)$, where x_0 is fixed and t varies in an interval), or else it can be realized as $t \mapsto (t, z(t))$, where $z(t)$ is a strictly concave function of t varying in an interval.

Let (x', z) be the first ("leftmost") point with second coordinate z on the geodesic arc from $(0, 0)$ to (x, z) in $\mathbb{H}(p)$, and let $(y', 0)$ be the last ("rightmost") point with second coordinate 0 on the geodesic arc from $(0, 0)$ to $(y, -z)$ in $\mathbb{H}(q)$. We may have $x' = x$ or $y' = 0$. The geodesic arc from $(0, z)$ to (x', z) in $\mathbb{H}(p)$ is strictly increasing in both coordinates, while the geodesic arc from $(y', 0)$ to $(y, -z)$ in $\mathbb{H}(q)$ is strictly increasing in the first and strictly decreasing in the second variable. That is, these two arcs can be parametrized, respectively, as

$$t \mapsto (x(t), t) \quad \text{and} \quad t \mapsto (y(t), -t),$$

where $t \in [0, z]$ and $\dot{x}(t), \dot{y}(t) > 0$. Now we can "synchronize" the two in order to get the curve

$$t \mapsto (x(t), y(t), t), \quad t \in [0, z],$$

that connects $(0, y', 0)$ with (x', y, z) in $\text{Sol}(p, q)$. The length of this curve majorises the distance between these two points in $\text{Sol}(p, q)$ and is

$$\begin{aligned} & \int_0^z \sqrt{e^{-2pt} \dot{x}(t)^2 + e^{2qt} \dot{y}(t)^2 + 1} dt \\ & \leq \int_0^z (\sqrt{e^{-2pt} \dot{x}(t)^2 + 1} + \sqrt{e^{2qt} \dot{y}(t)^2 + 1} - 1) dz \\ & = d_{\mathbb{H}(p)}((0, 0), (x', z)) + d_{\mathbb{H}(q)}((y', 0), (y, -z)) - z. \end{aligned}$$

Now, by (1.5),

$$d_{\text{Sol}}(o, \mathfrak{z}) \leq \underbrace{d_{\text{Sol}}((0, 0, 0), (0, y', 0))}_{=d_{\mathbb{H}(q)}((0, 0), (y', 0))} + d_{\text{Sol}}((0, y', 0), (x', y, z)) + \underbrace{d_{\text{Sol}}((x', y, z), (x, y, z))}_{=d_{\mathbb{H}(p)}((x', z), (x, z))}.$$

We insert the upper bound for the middle term that we derived above. Since

$$\begin{aligned} d_{\mathbb{H}(p)}((0, 0), (x', z)) + d_{\mathbb{H}(p)}((x', z), (x, z)) &= d_{\mathbb{H}(p)}((0, 0), (x, z)) \quad \text{and} \\ d_{\mathbb{H}(q)}((0, 0), (y', 0)) + d_{\mathbb{H}(q)}((y', 0), (y, -z)) &= d_{\mathbb{H}(q)}((0, 0), (y, -z)), \end{aligned}$$

the proposed inequality follows. For the second part of (iii), we use (2.3):

$$\begin{aligned} d_{\mathbb{H}(p)}((x, z), (0, 0)) &\leq |z| + \frac{1}{p} d_{\mathbb{H}(1)}((px, 0), (0, 0)) \\ &= |z| + \frac{1}{p} \log \frac{\sqrt{(px)^2 + 4} + |px|}{\sqrt{(px)^2 + 4} - |px|} \\ &\leq |z| + \frac{1}{p} \log((px)^2 + p|x| + 1) \leq |z| + \frac{2 \log(1 + p)}{p} + \frac{2 \log(1 + |x|)}{p}. \end{aligned}$$

Combining this with the analogous bound for $d_{\mathbb{H}(q)}((y, -z), (0, 0))$, the inequality follows.

For proving (iv), first note that for all $x \neq 0$,

$$d_{\mathbb{H}(p)}\left(\left(0, \frac{\log |x|}{p}\right), \left(x, \frac{\log |x|}{p}\right)\right) = c_p$$

depends only on p . Then, using (1.5),

$$\begin{aligned} d_{\text{Sol}(o, \mathfrak{z})} &\leq d_{\text{Sol}}\left(o, \left(0, 0, \frac{\log |x|}{p}\right)\right) + d_{\text{Sol}}\left(\left(0, 0, \frac{\log |x|}{p}\right), \left(x, 0, \frac{\log |x|}{p}\right)\right) \\ &\quad + d_{\text{Sol}}\left(\left(x, 0, \frac{\log |x|}{p}\right), \left(x, 0, -\frac{\log |y|}{q}\right)\right) \\ &\quad + d_{\text{Sol}}\left(\left(x, 0, -\frac{\log |y|}{q}\right), \left(x, y, -\frac{\log |y|}{q}\right)\right) + d_{\text{Sol}}\left(\left(x, y, -\frac{\log |y|}{q}\right), \mathfrak{z}\right) \\ &= \left|\frac{\log |x|}{p}\right| + c_p + \left|\frac{\log |x|}{p} + \frac{\log |y|}{q}\right| + c_q + \left|\frac{\log |y|}{q} + z\right|. \end{aligned}$$

Exchanging the roles of x and y , as well as of p and q , the inequality follows. ■

3 Central Limit Theorem and Rate of Escape

Let $\mathfrak{z}_t = (X_t, Y_t, Z_t)$, where $t \geq 0$, be the continuous diffusion on $\text{Sol}(p, q) \cong \mathbb{R}^3$ whose infinitesimal generator is \mathcal{L}_a . If the starting point is $o = (0, 0, 0)$, then \mathfrak{z}_t is given by the

stochastic integrals

$$\begin{aligned} Z_t &= a t + W_t, \\ X_t &= \int_0^t e^{pZ_s} dW_s^{(1)} \\ Y_t &= \int_0^t e^{-qZ_s} dW_s^{(2)}, \end{aligned} \tag{3.1}$$

where $(W_t, W_t^{(1)}, W_t^{(2)})_{t \geq 0}$ are three independent standard Brownian motions. (We do not attach a superscript to the one that defines the coordinate Z_t , because this is the most important one that determines the behavior of all three.) See, for instance, Revuz and Yor [41] or Protter [39], and compare, in particular, with Baldi et al. [5].

For the following central limit theorem, let

$$\mathcal{N} = W_1, \quad \underline{\mathcal{M}} = \min\{W_t : 0 \leq t \leq 1\} \quad \text{and} \quad \overline{\mathcal{M}} = \max\{W_t : 0 \leq t \leq 1\},$$

so that \mathcal{N} has standard normal distribution.

Theorem 3.1. (i) If $a > 0$, then as $t \rightarrow \infty$

$$\frac{1}{\sqrt{t}}(\log |X_t| - p a t, \log |Y_t|, Z_t - a t) \rightarrow (p \mathcal{N}, 0, \mathcal{N}) \quad \text{in law.}$$

(ii) If $a < 0$, then as $t \rightarrow \infty$

$$\frac{1}{\sqrt{t}}(\log |X_t|, \log |Y_t| + q a t, Z_t - a t) \rightarrow (0, q \mathcal{N}, \mathcal{N}) \quad \text{in law.}$$

(iii) If $a = 0$, then as $t \rightarrow \infty$

$$\frac{1}{\sqrt{t}}(\log |X_t|, \log |Y_t|, Z_t) \rightarrow (p \overline{\mathcal{M}}, -q \underline{\mathcal{M}}, \mathcal{N}) \quad \text{in law.} \quad \square$$

Proof. For $\alpha \in \mathbb{R}$, set

$$V_t(\alpha) = \int_0^t e^{2\alpha Z_s} ds,$$

so that the quadratic variations of X_t and Y_t are $V_t(p)$ and $V_t(-q)$, respectively. Then by a theorem of Dambis et al. [13, 17], see also [41, p. 173], there exist two standard Brownian

motions $(B_t^{(1)})_{t \geq 0}$ and $(B_t^{(2)})_{t \geq 0}$ such that

$$X_t = B_{V_t(\mathfrak{p})}^{(1)} \quad \text{and} \quad Y_t = B_{V_t(-\mathfrak{q})}^{(2)}. \quad (3.2)$$

By a theorem of Knight [32], see [41, p. 175], the processes $(B_t^{(1)})_{t \geq 0}$ and $(B_t^{(2)})_{t \geq 0}$ are independent in our case.

By the scaling property of Brownian motion, for $i = 1, 2$ and $\alpha = \mathfrak{p}$, resp. $\alpha = -\mathfrak{q}$,

$$\frac{\log |B_{V_t(\alpha)}^{(i)} / \sqrt{V_t(\alpha)}|}{\sqrt{t}} = \frac{\log |B_1|}{\sqrt{t}} \rightarrow 0 \quad \text{in law.}$$

In the following computations we use frequently the following simple fact.

$$\text{If } A_t \rightarrow A \text{ and } C_t \rightarrow 0 \text{ in law, then } (A_t, C_t) \rightarrow (A, 0) \text{ in law, as } t \rightarrow \infty. \quad (3.3)$$

Thus,

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} (\log |X_t| - \mathfrak{p} a t, \log |Y_t|, Z_t - a t) \\ &= \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} (\log \sqrt{V_t(\mathfrak{p})} - \mathfrak{p} a t, \log \sqrt{V_t(-\mathfrak{q})}, W_t) \quad \text{in law.} \end{aligned} \quad (3.4)$$

Case (i): $\mathfrak{a} > 0$. First observe that for all $\alpha > 0$ and $\beta \in \mathbb{R}$,

$$\lim_{t \rightarrow \infty} \int_0^t e^{-\alpha s + \beta W_s} ds = \int_0^\infty e^{-\alpha s + \beta W_s} ds \in (0, +\infty) \quad \text{almost surely,}$$

since $\lim_{s \rightarrow \infty} \log(e^{-\alpha s + \beta W_s})/s = -\alpha < 0$. Therefore

$$\frac{1}{\sqrt{t}} \log \sqrt{V_t(-\alpha)} \rightarrow 0 \quad \text{in law, when } \alpha > 0. \quad (3.5)$$

Using (3.3), we obtain (in law) that

$$\begin{aligned} & \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} (\log \sqrt{V_t(\mathfrak{p})} - \mathfrak{p} a t, \log \sqrt{V_t(-\mathfrak{q})}, W_t) \\ &= \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} (\log \sqrt{V_t(\mathfrak{p})} - \mathfrak{p} a t, 0, W_t) \end{aligned}$$

$$\begin{aligned}
&= \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \left(\frac{1}{2} \log \left(\int_0^t e^{2p(a(s-t)+W_s)} ds \right), 0, W_t \right) \\
&= \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \left(\frac{1}{2} \log \left(e^{2pW_t} \int_0^t e^{2p(a(s-t)+W_s-W_t)} ds \right), 0, W_t \right).
\end{aligned}$$

Since $(W_{t-s})_{s \leq t} = (W_t - W_s)_{s \leq t}$ in law,

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \log \left(\int_0^t e^{2p(a(s-t)+W_s-W_t)} ds \right) &= \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \log \left(\int_0^t e^{-2p(a(t-s)-W_{t-s})} ds \right) \\
&= \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \log \left(\int_0^t e^{-2p(a(s)-W_s)} ds \right) = 0
\end{aligned}$$

and using (3.3) once more, we get that

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} (\log |X_t| - p a t, \log |Y_t|, Z_t - a t) = \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} (p W_t, 0, W_t) = (p \mathcal{N}, 0, \mathcal{N})$$

in law.

Case (ii): $a < 0$. This is obtained from Case (i) by exchanging the roles of the x - and y -coordinates.

Case (iii): $a = 0$. We take up (3.4) and continue to compute, with all identities holding in law

$$\begin{aligned}
&\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} (\log \sqrt{V_t(p)}, \log \sqrt{V_t(-q)}, W_t) \\
&= \lim_{t \rightarrow \infty} \frac{1}{2\sqrt{t}} \left(\log \left(\int_0^t e^{2pW_s} ds \right), \log \left(\int_0^t e^{-2qW_s} ds \right), 2W_t \right) \\
&= \lim_{t \rightarrow \infty} \frac{1}{2\sqrt{t}} \left(\log \left(t \int_0^1 e^{2pW_{st}} ds \right), \log \left(t \int_0^1 e^{-2qW_{st}} ds \right), 2W_t \right) \\
&= \lim_{t \rightarrow \infty} \frac{1}{2\sqrt{t}} \left(\log \left(t \int_0^1 e^{2p\sqrt{t}W_s} ds \right), \log \left(t \int_0^1 e^{-2q\sqrt{t}W_s} ds \right), 2\sqrt{t}W_1 \right)
\end{aligned}$$

[setting $\tau = 2\sqrt{t}$]

$$= \lim_{\tau \rightarrow \infty} \left(\log \left(\int_0^1 e^{\tau p W_s} ds \right)^{1/\tau}, \log \left(\int_0^1 e^{-\tau q W_s} ds \right)^{1/\tau}, W_1 \right)$$

since $(W_{st})_{0 \leq s \leq 1} = (\sqrt{t}W_s)_{0 \leq s \leq 1}$ in law. Now recall that the L^τ -norm on $C([0, 1])$ converges to the L^∞ -norm as $\tau \rightarrow \infty$. We apply this to the functions $s \mapsto e^{pW_s}$ and $s \mapsto e^{-qW_s}$, respectively, and then take logarithms. Thus, almost surely

$$\begin{aligned} \log \left(\int_0^1 e^{\tau p W_s} ds \right)^{1/\tau} &\rightarrow p \cdot \max\{W_s : 0 \leq s \leq 1\} \quad \text{and} \\ \log \left(\int_0^1 e^{-\tau q W_s} ds \right)^{1/\tau} &\rightarrow -q \cdot \min\{W_s : 0 \leq s \leq 1\}. \end{aligned}$$

This leads to statement (iii). ■

Next, with \mathcal{N} , $\underline{\mathcal{M}}$, and $\bar{\mathcal{M}}$ as above, we deduce the following central limit theorem for the distance of Brownian motion to the origin.

Theorem 3.2. If $a \neq 0$, then

$$\frac{d_{\text{Sol}}(\mathfrak{J}_t, o) - |a|t}{\sqrt{t}} \rightarrow \mathcal{N} \text{ in law as } t \rightarrow \infty.$$

If $a = 0$, then

$$\frac{d_{\text{Sol}}(\mathfrak{J}_t, o)}{\sqrt{t}} \rightarrow 2(\bar{\mathcal{M}} - \underline{\mathcal{M}}) - |\mathcal{N}| \text{ in law as } t \rightarrow \infty. \quad \square$$

Proof. We start with $a > 0$. Combining Theorem 3.1(i) with Proposition 2.8(iv), we obtain in law

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{d_{\text{Sol}}(\mathfrak{J}_t, o) - at}{\sqrt{t}} &\leq \lim_{t \rightarrow \infty} \frac{\left| \frac{\log |X_t|}{p} + \frac{\log |Y_t|}{q} \right| + \left| \frac{\log |Y_t|}{q} \right| + \left| \frac{\log |X_t|}{p} - Z_t \right| - at}{\sqrt{t}} \\ &= \lim_{t \rightarrow \infty} \frac{\left| \frac{\log |X_t|}{p} \right| - at + \left| \left(\frac{\log |X_t|}{p} - at \right) - (Z_t - at) \right|}{\sqrt{t}} \\ &= \lim_{t \rightarrow \infty} \frac{\left| \frac{\log |X_t|}{p} \right| - at}{\sqrt{t}} \quad (\text{since } \Pr[\log |X_t| < 0] \rightarrow 0) \\ &= \lim_{t \rightarrow \infty} \frac{\frac{\log |X_t|}{p} - at}{\sqrt{t}} = \mathcal{N}. \end{aligned}$$

On the other hand,

$$\lim_{t \rightarrow \infty} \frac{d_{\text{Sol}}(\mathfrak{Z}_t, \circ) - \mathfrak{a}t}{\sqrt{t}} \geq \lim_{t \rightarrow \infty} \frac{|Z_t| - \mathfrak{a}t}{\sqrt{t}} = \mathcal{N} \quad \text{in law.}$$

When $\mathfrak{a} < 0$, the result follows once more by exchanging the roles of the x - and y -coordinates.

Now consider the case when $\mathfrak{a} = 0$. Combining Theorem 3.1(iii) with Proposition 2.8(iv), we obtain in law

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{d_{\text{Sol}}(\mathfrak{Z}_t, \circ)}{\sqrt{t}} &\leq |\bar{\mathcal{M}} - \underline{\mathcal{M}}| + \min\{|\bar{\mathcal{M}}| + |\mathcal{N} - \underline{\mathcal{M}}|, |\underline{\mathcal{M}}| + |\bar{\mathcal{M}} - \mathcal{N}|\} \\ &= 2(\bar{\mathcal{M}} - \underline{\mathcal{M}}) - |\mathcal{N}|. \end{aligned}$$

This upper bound together with the fact that $d_{\text{Sol}}(\mathfrak{Z}_t, \circ) \rightarrow \infty$ almost surely yields that

$$\frac{\log d_{\text{Sol}}(\mathfrak{Z}_t, \circ)}{\sqrt{t}} \rightarrow 0$$

in law (and in fact almost surely). We can combine this with Theorem 3.1(iii) and Proposition 2.8(i), and obtain the required lower bound in the case $\mathfrak{a} = 0$. ■

Compare this with the analogous result of [9] for simple random walk with drift on Diestel–Leader graphs. We next observe the following (denoting expectation by \mathbf{E}):

Lemma 3.3. Let $U_n = \max\{d_{\text{Sol}}(\mathfrak{Z}_n, \mathfrak{Z}_{n+t}) : 0 \leq t \leq 1\}$. Then $\mathbf{E}(U_n) < \infty$, and

$$\lim_{n \rightarrow \infty} \frac{1}{\log n} \max\{d_{\text{Sol}}(\mathfrak{Z}_n, \mathfrak{Z}_{n+t}) : 0 \leq t \leq 1\} = 0 \quad \text{almost surely.} \quad \square$$

Proof. The random variables U_n , $n \geq 0$, are i.i.d. Let

$$X_* = \max\{|X_t| : 0 \leq t \leq 1\}, \quad Y_* = \max\{|Y_t| : 0 \leq t \leq 1\}, \quad Z_* = \max\{|Z_t| : 0 \leq t \leq 1\}.$$

Then by Proposition 2.8(iii),

$$U_0 \leq \tilde{c} \times (1 + \log(1 + |X_*|) + \log(1 + |Y_*|) + |Z_*|).$$

Observe that by the Burkholder–Davis–Gundy inequality [41, p. 161], we have for every $r > 2$ that

$$\mathbb{E}(X_*^r) \leq c_1 \mathbb{E} \left(\left(\int_0^1 e^{2pZ_s} ds \right)^{r/2} \right) \leq c_1 \int_0^1 \mathbb{E}(e^{prZ_s}) ds < \infty,$$

where $c_1 > 0$. The same holds for Y_* . For Z_* , observe that by duality

$$\begin{aligned} \Pr[Z_* > z] &= \Pr[\max\{W_t - at : 0 \leq t \leq 1\} > z \text{ or } \max\{-W_t + at : 0 \leq t \leq 1\} > z] \\ &\leq 4\Pr[W_1 > z - |a|]. \end{aligned}$$

Thus,

$$\mathbb{E}(e^{rZ_*}) = \int_0^\infty \Pr[e^{rZ_*} > u] du \leq 4 \int_0^\infty \Pr[e^{r(W_1+|a|)} > u] du = 4 \mathbb{E}(e^{r(W_1+|a|)}) < \infty.$$

We find that for all $r > 0$,

$$\mathbb{E}(e^{rU_n}) = \mathbb{E}(e^{rU_0}) \leq e^{cr} \mathbb{E}((1 + X_*)^{cr}(1 + Y_*)^{cr} e^{crZ_*}) < \infty.$$

By the law of large numbers, $e^{rU_n}/n \rightarrow 0$ almost surely, whence $\limsup_{n \rightarrow \infty} U_n/\log n < 1/r$ almost surely, for all $r > 0$. ■

Given the (left) action on $\text{Sol}(p, q)$ on itself by isometries and the group-invariance of our Laplacian (Lemma 2.2), we obtain that along any time interval $[s, t]$, the increment of our Brownian motion $\mathfrak{J}_t = (X_t, Y_t, Z_t)$ of (3.1) satisfies

$$\mathfrak{J}_s^{-1} \mathfrak{J}_t = \mathfrak{J}_{t-s} \quad \text{in law,} \tag{3.6}$$

and for an arbitrary number of time intervals which do not overlap (i.e., they meet at most at the endpoints), the associated increments are independent. We now also get the rate of escape for our Brownian motion with drift.

Corollary 3.4. For any value of a ,

$$\lim_{t \rightarrow \infty} \frac{d_{\text{Sol}}(\mathfrak{J}_t, \mathfrak{o})}{t} = |a| \quad \text{almost surely.} \quad \square$$

Proof. In view of Lemma 3.3 and the spatial homogeneity (3.6), the subadditive ergodic theorem of Kingman [31] implies that $d_{\text{Sol}}(\mathfrak{J}_t, \mathfrak{o})/t$ converges almost surely to a constant.

(Compare with Derriennic [16] for the case of discrete time.) Theorem 3.2 implies that the limit is a in probability, whence also with probability 1. ■

4 Convergence to the Boundary at Infinity, and the Deviation from the Limit Geodesic

The natural geometric compactification of the hyperbolic plane, in the unit disk model, is just the closed (Euclidean) disk. In the upper half plane model $\mathbb{H}(p)$, the boundary at infinity $\partial\mathbb{H}(p)$ of the compactification $\widehat{\mathbb{H}}(p)$ is obtained by adding the bottom line $\partial^*\mathbb{H}(p) = \mathbb{R}$ and the point at infinity, denoted here by ϖ_p . In the logarithmic model, convergence to the boundary is as follows: we have that $(x, z) \rightarrow \xi \in \partial^*\mathbb{H}(p)$ when $z \rightarrow -\infty$ and $x \rightarrow \xi$, and $(x, z) \rightarrow \varpi_p$ if $|x| + e^z \rightarrow \infty$.

Now $\text{Sol}(p, q)$ embeds into $\mathbb{H}(p) \times \mathbb{H}(q)$ via (1.6). Therefore, the most natural geometric compactification $\widehat{\text{Sol}}(p, q)$ of $\text{Sol}(p, q)$ is its closure in the compact bidisk $\widehat{\mathbb{H}}(p) \times \widehat{\mathbb{H}}(q)$. (“Bidisk” because when we use the unit disk model of hyperbolic plane, this is just the direct product of two closed unit disks.) We assemble a brief description of convergence to the boundary in the next lemma; no proof is required. We underline once more the analogy with Diestel–Leader graphs [48] and treebolic space [8]. As pointed out in Section 1, the boundary at infinity is topologically a filled number “8”, that is, two closed disks glued together at a single point. This sheds some light on the observations made by Lyons and Sullivan [36].

Lemma 4.1. The boundary at infinity $\partial\text{Sol}(p, q)$ of $\text{Sol}(p, q)$ is

$$\left(\underbrace{\partial^*\mathbb{H}(p)}_{\mathbb{R}} \times \{\varpi_q\} \right) \cup \left(\{\varpi_p\} \times \underbrace{\partial^*\mathbb{H}(q)}_{\mathbb{R}} \right) \cup (\mathbb{H}(p) \times \{\varpi_q\}) \cup (\{\varpi_p\} \times \mathbb{H}(q)) \cup \{(\varpi_p, \varpi_q)\}.$$

Convergence to the boundary is as follows. In general,

$$r = (x, y, z) \rightarrow (\xi, \eta) \in \partial\text{Sol}(p, q) \quad \text{if } (x, z) \rightarrow \xi \text{ in } \widehat{\mathbb{H}}(p) \text{ and } (y, -z) \rightarrow \eta \text{ in } \widehat{\mathbb{H}}(q).$$

This means that

$$r = (x, y, z) \rightarrow (\xi, \varpi_q) \in \partial^*\mathbb{H}(p) \times \{\varpi_q\} \quad \text{if } x \rightarrow \xi \text{ and } z \rightarrow -\infty,$$

$$r = (x, y, z) \rightarrow (\varpi_p, \eta) \in \{\varpi_p\} \times \partial^*\mathbb{H}(q) \quad \text{if } y \rightarrow \eta \text{ and } z \rightarrow \infty,$$

$$r = (x, y, z) \rightarrow ((x_0, z_0), \varpi_q) \in \mathbb{H}(p) \times \{\varpi_q\} \quad \text{if } x \rightarrow x_0, z \rightarrow z_0 \text{ and } |y| \rightarrow \infty,$$

$$\mathfrak{r} = (x, y, z) \rightarrow (\varpi_p, (y_0, -z_0)) \in \{\varpi_p\} \times \mathbb{H}(\mathfrak{q}) \quad \text{if } y \rightarrow y_0, z \rightarrow -z_0 \text{ and } |x| \rightarrow \infty,$$

$$\mathfrak{r} = (x, y, z) \rightarrow (\varpi_p, \varpi_q) \quad \text{if } |x| + e^z \rightarrow \infty \text{ and } |y| + e^{-z} \rightarrow \infty \text{ in } \mathbb{R}. \quad \square$$

A *geodesic ray* is a continuous mapping $\gamma : [0, \infty) \rightarrow \text{Sol}$ (or to any of our other spaces) such that $d(\gamma(t), \gamma(s)) = |t - s|$ for all s, t . Its starting point is $\gamma(0)$. For any $(x_0, z_0) \in \mathbb{H}(\mathfrak{p})$ and $\xi \in \partial\mathbb{H}(\mathfrak{p})$, there is a unique geodesic ray $(x(t), z(t))$ that starts at (x_0, z_0) and converges to ξ . In the case when $\xi = \varpi_p$ then this is the upwards going vertical half-line $t \mapsto (x_0, z_0 + t)$ in $\mathbb{H}(\mathfrak{p})$.

For $\mathfrak{r} = (x_0, y_0, z_0) \in \text{Sol}(\mathfrak{p}, \mathfrak{q})$ and a boundary point $(\varpi_p, \eta) \in \{\varpi_p\} \times \partial^*\mathbb{H}(\mathfrak{q})$ of $\text{Sol}(\mathfrak{p}, \mathfrak{q})$, we can consider the (unique) upwards geodesic ray starting at \mathfrak{r} given by $\gamma_{\mathfrak{r}}^{\eta}(t) = (x_0, y(t), z(t))$, where $(y(t), -z(t))_{t \geq 0}$ is the geodesic ray from $(y_0, -z_0)$ to η in $\mathbb{H}(\mathfrak{q})$.

Analogously, for a boundary point $(\xi, \varpi_q) \in \partial^*\mathbb{H}(\mathfrak{p}) \times \{\varpi_q\}$, we have the (unique) downwards geodesic ray starting at \mathfrak{r} given by $\gamma_{\mathfrak{r}}^{\xi}(t) = (x(t), y_0, z(t))$, where $(x(t), z(t))_{t \geq 0}$ is the geodesic ray from (x_0, z_0) to ξ in $\mathbb{H}(\mathfrak{p})$. All those geodesics converge to their defining boundary points, as $t \rightarrow \infty$, and any two geodesics that converge to the same boundary point are at bounded Hausdorff distance. This is true because it holds in the hyperbolic plane.

In the first of the above two cases, it will be most convenient to use the initial point $\mathfrak{r} = (0, \eta, 0)$, and omit the index \mathfrak{r} in that case. Thus, we can parameterize by $z \geq 0$ and get $\gamma^{\eta}(z) = (0, \eta, z)$. Analogously, in the second case, we use the standard initial point $\mathfrak{r} = (\xi, 0, 0)$ and obtain the corresponding geodesic ray $\gamma^{\xi}(z) = (\xi, 0, -z)$, again parameterized by $z \geq 0$. We call these the (upwards, resp. downwards) *vertical* geodesic rays. We remark that there is no geodesic ray in Sol from any starting point that converges to (ϖ_p, ϖ_q) .

Compare with [19, 20] for further details on the geometry.

Let us return to our Brownian motion $\mathfrak{Z}_t = (X_t, Y_t, Z_t)$ of (3.1).

Proposition 4.2. (i) If $a > 0$, then

$$\lim_{t \rightarrow \infty} Y_t = Y_{\infty} = \int_0^{\infty} e^{-qZ_s} dW_s^{(2)} \quad \text{almost surely.}$$

That is, $\mathfrak{Z}_t \rightarrow (\varpi_p, Y_{\infty}) \in \partial\text{Sol}(\mathfrak{p}, \mathfrak{q})$ almost surely in the topology of $\widehat{\text{Sol}}(\mathfrak{p}, \mathfrak{q})$. (ii) If $a < 0$, then

$$\lim_{t \rightarrow \infty} X_t = X_{\infty} = \int_0^{\infty} e^{pZ_s} dW_s^{(1)} \quad \text{almost surely.}$$

That is, $\mathfrak{Z}_t \rightarrow (X_{\infty}, \varpi_q)$ almost surely in the topology of $\widehat{\text{Sol}}(\mathfrak{p}, \mathfrak{q})$.

In both cases (i) and (ii), the respective limiting random variable is almost surely finite.

(iii) If $a = 0$, then

$$|X_t| + e^{Z_t} \rightarrow \infty \quad \text{and} \quad |Y_t| + e^{-Z_t} \rightarrow \infty \quad \text{almost surely.}$$

That is, $\mathfrak{Z}_t \rightarrow (\varpi_p, \varpi_q)$ almost surely in the topology of $\widehat{\text{Sol}}(p, q)$. \square

Proof. (i) and (ii) are immediate from the representation (3.1) via [41, Proposition 1.26].

For (iii), consider $\mathfrak{X}_t = (X_t, Z_t)$ as a process on the affine group $\text{Aff}(p)$ of (2.1).

It also satisfies (3.6) and (Lemma 3.3). We consider our process at discrete times:

$$\mathfrak{X}_n = (X_n, Z_n) = \mathfrak{X}_1 \cdot (\mathfrak{X}_1^{-1} \mathfrak{X}_2) \cdots (\mathfrak{X}_{n-1}^{-1} \mathfrak{X}_n) \quad (4.1)$$

is a right random walk on $\text{Aff}(p)$. We can apply a result of Brofferio [10]. In the notation of [10], $A_1 = e^{pZ_1}$ and $B_1 = X_1$. Since the expectation of $\log A_1$ is 0, and all moment conditions of [10, Theorem 1] are satisfied, $\mathfrak{X}_n \rightarrow \varpi_p$ almost surely in $\mathbb{H}(p)$, as $n \rightarrow \infty$ in \mathbb{Z} . By Lemma 3.3, also $\mathfrak{X}_t \rightarrow \varpi_p$ almost surely, as $t \rightarrow \infty$ in \mathbb{R} .

In the same way, $(Y_t, -Z_t) \rightarrow \varpi_q$ almost surely, as $t \rightarrow \infty$ in \mathbb{R} . Statement (iii) follows. \blacksquare

Remark 4.3. When $a > 0$, the distribution of the random variable Y_∞ is well understood; compare with [5]. Indeed, by (3.2),

$$Y_\infty = B_{V_\infty(-q)}^{(2)} \stackrel{\text{in law}}{=} \sqrt{V_\infty(-q)} B_1^{(2)},$$

and the latter is a product of two independent random variables. Now

$$4q^2 V_\infty(-q) \stackrel{\text{in law}}{=} 4q^2 \int_0^\infty e^{-2qa t - W_{4q^2 t}} dt \stackrel{\text{in law}}{=} \int_0^\infty e^{-\frac{a}{2q} s - W_s} ds$$

is Dufresne's integral with $c = a/2q$. By the formula of Dufresne [18]—see also Bailleul [3]—the density of $\sqrt{V_\infty(-q)}$ is computed as

$$u \mapsto \frac{2}{(2q^2)^\kappa \Gamma(\kappa/2)} \times \frac{e^{-1/(2q^2 u^2)}}{u^{1+\kappa}}, \quad u > 0, \quad \text{where } \kappa = \kappa(q) = 2|a|/q.$$

The density $f_\infty = f_\infty^{(a,q)}$ of Y_∞ can now be obtained as the multiplicative convolution of this function with the standard Gaussian density:

$$f_\infty(y) = \binom{(\kappa - 1)/2}{\frac{1}{2}} \frac{q}{2(1 + q^2 y^2)^{(\kappa+1)/2}}, \quad y \in \mathbb{R}.$$

The same applies of course to X_∞ when $a < 0$, with p in the place of q . □

Thus, when $a > 0$, we have the vertical *limit geodesic* γ^{Y_∞} to whose limit point our Brownian motion converges, and when $a < 0$ we have to replace this by γ^{X_∞} . In order to simplify notation, we just write γ^∞ for the respective limit geodesic in each of those cases.

We now prove that when $a \neq 0$, the convergence of \mathfrak{Z}_t to its boundary limit is very straight, in the sense that its deviation from γ^∞ is of the order of $\log t$.

Theorem 4.4. If $a \neq 0$, then Brownian motion on $\text{Sol}(p, q)$ satisfies

$$\limsup_{t \rightarrow \infty} \frac{1}{\log t} d(\mathfrak{Z}_t, \gamma^\infty) \leq \frac{2}{|a|} \quad \text{almost surely,}$$

where $d(\mathfrak{Z}_t, \gamma^\infty) = \inf\{d(\mathfrak{Z}_t, \gamma^\infty(u)) : u \geq 0\}$. □

Proof. Once more, it is sufficient to consider only the case $a > 0$.

For each $t \geq 0$, the point $(0, Y_\infty, Z_t)$ lies on the geodesic $\gamma^\infty = \gamma^{Y_\infty}$. We shall show that for integer n ,

$$\limsup_{n \rightarrow \infty} \frac{1}{\log n} d(\mathfrak{Z}_n, (0, Y_\infty, Z_n)) \leq \frac{2}{a} \quad \text{almost surely,} \tag{4.2}$$

Together with Lemma 3.3, this will yield the result.

The metric d_{Sol} is invariant under the left action of the group $\text{Sol}(p, q)$. Using the product formula (1.3) and subsequently Proposition 2.8(iii), we find

$$\begin{aligned} d_{\text{Sol}}(\mathfrak{Z}_t, (0, Y_\infty, Z_t)) &= d_{\text{Sol}}((e^{-pZ_t} X_t, e^{qZ_t}(Y_t - Y_\infty), 0), o) \\ &\leq c + \frac{2}{p} \log(1 + e^{-pZ_t}|X_t|) + \frac{2}{q} \log(1 + e^{qZ_t}|Y_t - Y_\infty|). \end{aligned}$$

We have

$$\begin{aligned} e^{qZ_t}(Y_\infty - Y_t) &= e^{q(W_t+at)} \int_t^\infty e^{-q(W_s+as)} dW_s^{(2)} = \int_t^\infty e^{-q((W_s-W_t)+a(s-t))} dW_s^{(2)} \\ &\stackrel{\text{in law}}{=} \int_0^\infty e^{-q(W_s+as)} dW_s^{(2)} = Y_\infty. \end{aligned}$$

By Remark 4.3, $\Pr[|Y_\infty| > y] \asymp y^{-\kappa(q)}$ as $y \rightarrow \infty$. Now take $\delta > 1/\kappa(q)$. Then

$$\begin{aligned} \sum_{n=2}^\infty \Pr[\log(1 + e^{qZ_n}|Y_n - Y_\infty|) > \delta \log n] &= \sum_{n=2}^\infty \Pr[\log(1 + |Y_\infty|) > \delta \log n] \\ &= \sum_{n=2}^\infty \Pr[|Y_\infty| > n^\delta - 1] \asymp \sum_{n=2}^\infty (n^\delta - 1)^{-\kappa(q)} < +\infty. \end{aligned}$$

Thus by the Borel–Cantelli Lemma

$$\limsup_{n \rightarrow \infty} \frac{1}{\log n} \log(1 + e^{qZ_n}|Y_n - Y_\infty|) \leq \frac{q}{2a} \quad \text{almost surely.}$$

We now consider the first coordinate. For fixed t observe that

$$\begin{aligned} e^{-pZ_t} X_t &= e^{-p(W_t+at)} \int_0^t e^{p(W_s+as)} dW_s^{(1)} \\ &= \int_0^t e^{-p((W_t-W_s)+a(t-s))} dW_s^{(1)} \\ &\stackrel{\text{in law}}{=} \int_0^t e^{-p(W_s+as)} dW_s^{(1)} =: \tilde{X}_t, \end{aligned}$$

and $\tilde{X}_\infty = \lim_{t \rightarrow \infty} \tilde{X}_t$ exists almost surely. As above, one finds that

$$e^{pZ_t}(\tilde{X}_\infty - \tilde{X}_t) = \int_t^\infty e^{-p((W_s-W_t)+a(s-t))} dW_s^{(1)} =: \bar{X}_{t,\infty},$$

where $\bar{X}_{t,\infty}$ is independent from $(\mathfrak{F}_s)_{0 \leq s \leq t}$ and has the same law as \tilde{X}_∞ . Thus, for some constant $C > 0$ and for any $x > 0$,

$$\begin{aligned} \Pr[e^{-pZ_n}|X_n| > x] &= \Pr\left[|\tilde{X}_n| > x, |\tilde{X}_\infty - \tilde{X}_n| \leq \frac{x}{2}\right] + \Pr\left[|\tilde{X}_n| > x, |\tilde{X}_\infty - \tilde{X}_n| > \frac{x}{2}\right] \\ &\leq \Pr\left[|\tilde{X}_\infty| > \frac{x}{2}\right] + \Pr\left[e^{-pZ_n}|\tilde{X}_{n,\infty}| > \frac{x}{2}\right] \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\frac{X}{2}\right)^{-\kappa(p)} + \mathbb{E} \left(\Pr \left[|\bar{X}_{n,\infty}| > \frac{X}{2} e^{\rho Z_n} \mid Z_n \right] \right) \\
&\leq C \left(\frac{X}{2}\right)^{-\kappa(p)} + C \mathbb{E} \left(\left(\frac{X}{2} e^{\rho Z_n} \right)^{-\kappa(p)} \right) \\
&\leq C \left(\frac{X}{2}\right)^{-\kappa(p)} + C \left(\frac{X}{2}\right)^{-\kappa(p)} \underbrace{\mathbb{E}(e^{-\kappa(p)\rho Z_1})}_=1^n = 2C \left(\frac{X}{2}\right)^{-\kappa(p)}
\end{aligned}$$

Proceeding as above, the Borel–Cantelli Lemma implies that

$$\limsup_{n \rightarrow \infty} \frac{1}{\log n} \log(1 + e^{-\rho Z_n} |X_n|) \leq \frac{\rho}{2a}$$

almost surely. ■

As mentioned in Section 1, the analogous result for Brownian motion with nonzero vertical drift on the hyperbolic plane is a special case of the result stated in [1, Théorème 7.3]. The latter is based on subtle potential theoretic arguments. By combining this with properties of the metric of $\text{Sol}(\rho, q)$, one can also deduce the qualitative statement of Theorem 4.4. Our proof uses much more direct computations and leads to the explicit bound $2/|a|$ for the log-deviation.

5 Elements of Potential Theory

If \mathcal{L} is any of our different Laplacians on Sol , \mathbb{H} , or \mathbb{R} , and $\lambda \in \mathbb{R}$, then we denote by $\mathcal{H}(\mathcal{L}, \lambda)$ the space of all functions h on our space which satisfy $\mathcal{L}h = \lambda \cdot h$. The positive cone $\mathcal{H}^+(\mathcal{L}, \lambda)$ contains nonzero functions if and only if $\lambda \geq \lambda_{\min}(\mathcal{L})$, the *bottom of the positive spectrum*. Below we shall clarify what the values of λ_{\min} are in each of our cases. In any case, $\lambda_{\min} \leq 0$, since the space of *harmonic functions* $\mathcal{H}(\mathcal{L}) = \mathcal{H}(\mathcal{L}, 0)$ contains all constant functions.

By the *minimum principle*, every nonzero function in $\mathcal{H}^+(\mathcal{L}, \lambda)$ must be strictly positive in each point.

A function h in $\mathcal{H}^+(\mathcal{L}, \lambda)$ is called *minimal* if $h(0) = 1$ and whenever $h \geq f \in \mathcal{H}^+(\mathcal{L}, \lambda)$ then f/h is constant. A basic fact in classical potential theory of Riemannian manifolds says that every function in $\mathcal{H}^+(\mathcal{L}, \lambda)$ can be expressed uniquely as an integral over the minimal harmonic functions with respect to a finite Borel measure on the latter set.

We shall specify this in more detail in our cases below.

Let us now recall what happens in the case of the standard Laplacian

$$\mathfrak{L}_{-1/2}^{\mathbb{H}} = \frac{1}{2} \left(e^{2z} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z} \right)$$

on standard hyperbolic plane $\mathbb{H} = \mathbb{H}(1) = \{x + i e^z : x, z \in \mathbb{R}\}$ in the logarithmic model.

The minimal harmonic functions are the *Poisson kernels*, which are parameterized by the (hyperbolic) boundary $\vartheta \mathbb{H} = \mathbb{R} \cup \{\varpi\}$. (Recall that $\varpi = \varpi_1$ is the point at infinity.) In the logarithmic model the kernels are

$$P((x, z), \varpi) = e^z \quad \text{and} \quad P((x, z), \xi) = \frac{(\xi^2 + 1)e^z}{(\xi - x)^2 + e^{2z}}, \quad \xi \in \mathbb{R}. \quad (5.1)$$

We have $\lambda_{\min}(\mathfrak{L}_{-1/2}^{\mathbb{H}}) = -\frac{1}{8}$, and the minimal elements in $\mathcal{H}^+(\mathfrak{L}_{-1/2}^{\mathbb{H}}, \lambda)$ are the functions

$$P(\cdot, \xi)^{\alpha(\lambda)} \quad \text{where } \xi \in \vartheta \mathbb{H} \quad \text{and} \quad \alpha(\lambda) = \frac{1 + \sqrt{1 + 8\lambda}}{2}. \quad (5.2)$$

Next, let us turn our attention to $\mathcal{H}^+(\mathfrak{L}_a^{\mathbb{H}(p)}, \lambda)$.

Lemma 5.1. A function f on $\mathbb{H}(p)$ is in $\mathcal{H}(\mathfrak{L}_a^{\mathbb{H}(p)}, \lambda)$ if and only if the function on $\mathbb{H}(1)$ given by $(e^{(a+p/2)z} f) \circ \theta$ is in $\mathcal{H}(\mathfrak{L}_{-1/2}^{\mathbb{H}(1)}, \frac{8\lambda + 4a^2 - p^2}{8p^2})$, where $\theta(x, z) = (x/p, z/p)$. In particular,

$$\lambda_{\min}(\mathfrak{L}_a^{\mathbb{H}(p)}) = -a^2/2,$$

and for $\lambda \geq -a^2/2$, the minimal elements in $\mathcal{H}^+(\mathfrak{L}_a^{\mathbb{H}(p)}, \lambda)$ are the functions

$$P_{p,a,\lambda}((x, z), \varpi_p) = e^{\alpha z} \quad \text{and} \quad P_{p,a,\lambda}((x, z), \xi) = e^{\alpha z} \left(\frac{(\xi^2 + 1)}{(\xi - px)^2 + e^{2pz}} \right)^{\beta}, \quad \xi \in \vartheta^* \mathbb{H}(p) \equiv \mathbb{R},$$

where $\alpha = \alpha(\lambda, a) = \sqrt{a^2 + 2\lambda} - a$ and $\beta = \beta(\lambda, a, p) = \frac{1}{2} + \frac{\sqrt{a^2 + 2\lambda}}{p}$. □

Proof. First of all, it is a straightforward computation that

$$(\mathfrak{L}_a^{\mathbb{H}(p)} f) \circ \theta = p^2 \mathfrak{L}_{a/p}^{\mathbb{H}(1)} (f \circ \theta).$$

Therefore, f is in $\mathcal{H}(\mathfrak{L}_a^{\mathbb{H}(p)}, \lambda)$ if and only if $\bar{f} = f \circ \theta$ is in $\mathcal{H}(\mathfrak{L}_{a/p}^{\mathbb{H}(1)}, \lambda/p^2)$.

For the moment, set $\bar{a} = a/p$ and $\bar{\lambda} = \lambda/p^2$. Then we compute

$$\mathfrak{L}_{-1/2}^{\mathbb{H}(1)}(e^{(\bar{a}+1/2)z} \bar{f}) = e^{(\bar{a}+1/2)z} \left(\mathfrak{L}_{\bar{a}}^{\mathbb{H}(1)} \bar{f} + \frac{4\bar{a}^2 - 1}{8} \bar{f} \right).$$

Therefore, \bar{f} is in $\mathcal{H}(\mathfrak{L}_{\bar{a}}^{\mathbb{H}(1)}, \bar{\lambda})$ if and only if $e^{(\bar{a}+1/2)z} \bar{f}$ is in $\mathcal{H}(\mathfrak{L}_{-1/2}^{\mathbb{H}(1)}, \bar{\lambda} + \frac{4\bar{a}^2-1}{8})$.

Combining these computations, the statements follow. ■

Thus, every function $h \in \mathcal{H}^+(\mathfrak{L}_a^{\mathbb{H}(p)}, \lambda)$ has a unique integral representation

$$h(x, z) = \int_{\mathfrak{H}} P_{p,a,\lambda}((x, z), \xi) \, d\nu(\xi), \tag{5.3}$$

where ν is a (finite, positive) Borel measure on \mathfrak{H} . (This includes $\xi = \varpi_p$.)

Remark 5.2. When $\lambda = 0$, that is, when we consider ordinary harmonic functions, we see that the constant function $\mathbf{1}$ is minimal harmonic if and only if $a \geq 0$. This can be stated also by saying that $\mathfrak{L}_a^{\mathbb{H}(p)}$ has the (weak) Liouville property, that is, all bounded harmonic functions are constant, precisely when $a \geq 0$. □

Everything that we have said so far in this section is very well known; see, for example, Helgason [25], or many other sources.

Let us now turn our attention to Sol. The following is immediate.

Lemma 5.3. If the function h_1 on $\mathbb{H}(p)$ is such that $h = h_1 \circ \pi_1$ is minimal in $\mathcal{H}^+(\mathfrak{L}_a^{\text{Sol}(p,q)}, \lambda)$, then h_1 is also minimal in $\mathcal{H}^+(\mathfrak{L}_a^{\mathbb{H}(p)}, \lambda)$.

In the same way, if the function h_2 on $\mathbb{H}(q)$ is such that $h = h_2 \circ \pi_2$ is minimal in $\mathcal{H}^+(\mathfrak{L}_a^{\text{Sol}(p,q)}, \lambda)$, then h_2 is also minimal in $\mathcal{H}^+(\mathfrak{L}_{-a}^{\mathbb{H}(q)}, \lambda)$. □

We now need a part of the Martin boundary theory for elliptic operators on manifolds. The reader is referred to Ancona [1] and Taylor [44] for the necessary background material. See also [24, Chapter VI]. In the following propositions, we subsume the necessary material without all proofs.

Proposition 5.4. The Markov semigroup $\mathfrak{H}_t = \mathfrak{H}_t^a = \exp(t\mathfrak{L}_a)$, $t > 0$, admits a symmetric, bounded kernel $\mathbf{h}_t(x, y) = \mathbf{h}_t^a(x, y)$ with respect to the measure \mathbf{m}_a of Lemma 2.6(a), such

that

$$\mathfrak{H}_t f(\mathfrak{r}) = \int_{\text{Sol}} \mathbf{h}_t(\mathfrak{r}, \mathfrak{z}) f(\mathfrak{z}) \, d\mathbf{m}_a(\mathfrak{z}).$$

For each $\mathfrak{z} = (x, y, z) \in \text{Sol}$, the function $\mathbf{h}_t(\cdot, \mathfrak{z})$ is in $C^2(\text{Sol})$. Furthermore, its kernel with respect to the volume element $d\mathfrak{z}$ of the Sol -manifold,

$$\mathbf{p}_t(\mathfrak{r}, \mathfrak{z}) = \mathbf{h}_t(\mathfrak{r}, \mathfrak{z}) e^{(a+p-q)z},$$

is stochastic and invariant under the action of the group $\text{Sol}(p, q)$. □

Proposition 5.5. The associated Green kernel

$$\mathbf{g}_a(\mathfrak{r}, \mathfrak{z}|\lambda) = \mathbf{g}(\mathfrak{r}, \mathfrak{z}|\lambda) = \int_0^\infty e^{-\lambda t} \mathbf{p}_t(\mathfrak{r}, \mathfrak{z}) \, dt \quad (\mathfrak{r}, \mathfrak{z} \in \text{Sol}, \mathfrak{r} \neq \mathfrak{z})$$

is strictly positive and finite for each $\lambda \geq \lambda_{\min}(\mathfrak{L}_a)$. □

We remark that finiteness at $\lambda = \lambda_{\min}(\mathfrak{L}_a)$ can be seen in different ways.

(1) It is well known that the Green kernel of $\mathfrak{L}_a^{\mathbb{H}(p)}$ exists (i.e., it is finite off the diagonal) at $\lambda = \lambda_{\min}$, see, for example, Korányi and Taylor [33, Proposition 1.2]. Via the projection π_1 , that kernel is obtained by integrating $\mathbf{g}_a(\mathfrak{r}, \mathfrak{z}|\lambda_{\min})$ over the y -variable of \mathfrak{z} . Thus, $\mathbf{g}_a(\mathfrak{r}, \mathfrak{z}|\lambda_{\min})$ must also be finite.

(2) It is a general fact known from potential theory that in the case when $\mathbf{g}_a(\cdot, \cdot|\lambda_{\min}) \equiv \infty$, the cone of positive eigenfunctions must be one-dimensional. This is not the case here, since we already have distinct positive eigenfunctions with value 1 at the origin on $\mathbb{H}(p)$, as one can see from Lemma 5.1 combined with (5.1) and (5.2).

(3) One may also invoke an observation of Guivarc’h [23, p. 85], formulated for random walks. It implies, in particular, that on a connected Lie group like ours the Green kernel at λ_{\min} can be infinite only when that group carries a recurrent random walk, which by Baldi [4] can only happen if the group contains a co-compact lattice which is Abelian of rank 1 or 2.

Proposition 5.6. For each $d > 0$ and $\lambda \geq \lambda_{\min}$, the Green kernel satisfies the Harnack inequality

$$\frac{\mathbf{g}(\mathfrak{r}, \mathfrak{z}'|\lambda)}{\mathbf{g}(\mathfrak{r}, \mathfrak{z}|\lambda)} \leq C_d(\lambda) \quad \text{and} \quad \frac{\mathbf{g}(\mathfrak{z}', \mathfrak{r}|\lambda)}{\mathbf{g}(\mathfrak{z}, \mathfrak{r}|\lambda)} \leq C_d(\lambda),$$

whenever $d(\mathfrak{z}, \mathfrak{z}') \leq d$ and $\min\{d(\mathfrak{z}, \mathfrak{r}), d(\mathfrak{z}', \mathfrak{r})\} \geq 10(d+1)$, where $C_d(\lambda) > 1$ is such that $C_d(\lambda) \rightarrow 1$ when $d \rightarrow 0$.

Furthermore, every function h in $\mathcal{H}^+(\mathfrak{L}_a^{\text{Sol}(p,q)}, \lambda)$ satisfies

$$\frac{h(\mathfrak{z}')}{h(\mathfrak{z})} \leq C_d(\lambda) \quad \text{for all } \mathfrak{z}, \mathfrak{z}' \in \text{Sol with } d(\mathfrak{z}, \mathfrak{z}') \leq d. \quad \square$$

Proof (outline). In the case \mathfrak{L}_a is the Laplace–Beltrami operator of $\text{Sol}(p, q)$, one can apply well-known Harnack inequalities of Li and Yau, see [35], because the Riemannian structure is invariant under a group action and thus the Ricci curvature is globally bounded below.

For arbitrary values of \mathfrak{a} , our operator is obtained by adding to the Laplace–Beltrami operator a multiple of $\frac{\partial}{\partial z}$, which leads just to conjugating our functions with an exponential in z , compare with the proof of Lemma 5.1. Thus, the inequalities hold with any drift term \mathfrak{a} . ■

The *Martin kernel* is

$$\mathbf{k}_a(\mathfrak{r}, \mathfrak{z}|\lambda) = \mathbf{k}(\mathfrak{r}, \mathfrak{z}|\lambda) = \frac{\mathfrak{g}_a(\mathfrak{r}, \mathfrak{z}|\lambda)}{\mathfrak{g}_a(0, \mathfrak{z}|\lambda)}, \quad \mathfrak{z} \neq 0, \mathfrak{r}. \quad (5.4)$$

The *Martin compactification* is the smallest compactification of the underlying space Sol (i.e., a Hausdorff space into which Sol embeds homeomorphically and densely) such that each function $\mathbf{k}_a(\mathfrak{r}, \cdot|\lambda)$ has a continuous extension in the second variable. The *Martin boundary* $\mathcal{M}(\lambda) = \mathcal{M}(\mathfrak{L}_a, \lambda)$ is the ideal boundary added to the space in that compactification. The extended kernel is also denoted by $\mathbf{k}_a(\cdot, \cdot|\lambda)$

Proposition 5.7. Every minimal eigenfunction h in $\mathcal{H}^+(\mathfrak{L}_a^{\text{Sol}(p,q)}, \lambda)$, $\lambda \geq \lambda_{\min}$, is of the form

$$h(\mathfrak{r}) = \mathbf{k}_a(\mathfrak{r}, \zeta|\lambda) \quad \text{where } \zeta \in \mathcal{M}(\lambda).$$

That is, there is a (suitable) sequence (\mathfrak{z}_n) in Sol with $d(0, \mathfrak{z}_n) \rightarrow \infty$, such that

$$h(\mathfrak{r}) = \lim_{n \rightarrow \infty} \mathbf{k}(\mathfrak{r}, \mathfrak{z}_n|\lambda). \quad \square$$

The *minimal Martin boundary* $\mathcal{M}_{\min}(\lambda) = \mathcal{M}_{\min}(\mathfrak{L}_a, \lambda)$ consists of all $\zeta \in \mathcal{M}(\lambda)$ for which $\mathbf{k}_a(\cdot, \zeta|\lambda)$ is minimal. It is a Borel subset of $\mathcal{M}(\lambda)$. The *Poisson–Martin representation theorem* says the following.

Proposition 5.8. For every function $h \in \mathcal{H}^+(\mathcal{L}_a^{\text{Sol}(p,q)}, \lambda)$, there is a unique Borel measure ν^h on $\mathcal{M}_{\min}(\lambda)$ such that

$$h(\mathfrak{r}) = \int_{\mathcal{M}_{\min}(\lambda)} \mathbf{k}(\mathfrak{r}, \cdot | \lambda) d\nu^h \quad \text{for every } \mathfrak{r} \in \text{Sol}. \quad \square$$

All this is of course true for more general manifolds and elliptic operators; see [44].

While we are not able to determine the whole Martin compactification, that is, the directions of convergence of the Martin kernels, we shall determine precisely the minimal positive λ -eigenfunctions for each $\lambda \geq \lambda_{\min}$.

6 Positive Harmonic Functions on $\text{Sol}(p, q)$

We now show that every positive eigenfunction of our Laplacian on $\text{Sol}(p, q)$ splits as a sum of two eigenfunctions that live on the two respective hyperbolic planes which make up Sol . The first step is the following.

Theorem 6.1. Let $h \in \mathcal{H}^+(\mathcal{L}_a^{\text{Sol}(p,q)}, \lambda)$ be minimal, where $\lambda \geq -a^2/2$.

Then $h(x, y, z) = h_1(x, z)$, where h_1 is minimal in $\mathcal{H}^+(\mathcal{L}_a^{\text{H}(p)}, \lambda)$, or $h(x, y, z) = h_2(y, -z)$, where h_2 is minimal in $\mathcal{H}^+(\mathcal{L}_{-a}^{\text{H}(q)}, \lambda)$. \square

Proof. Let h be a minimal eigenfunction in $\mathcal{H}^+(\mathcal{L}_a, \lambda)$. Then $h = \lim_{n \rightarrow \infty} \mathbf{k}(\cdot, \mathfrak{z}_n | \lambda)$. Write $\mathfrak{z}_n = (x_n, y_n, z_n)$.

Claim. (a) If $\inf_n z_n > -\infty$, then for each $a \in \mathbb{R}$,

$$h(x + a, y, z) = h(x, y, z) \quad \text{for all } (x, y, z) \in \text{Sol}.$$

(b) If $\sup_n z_n < +\infty$, then for each $b \in \mathbb{R}$,

$$h(x, y + b, z) = h(x, y, z) \quad \text{for all } (x, y, z) \in \text{Sol}. \quad \square$$

To prove part (a) of this claim, let $a \in \mathbb{R}$ and consider the group element

$$\mathfrak{g}_a = (a, 0, 0) = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{Sol}(p, q).$$

We abbreviate $\tau_a = \tau_{g_a}$. Let $\mathfrak{r} = (x, y, z) \in \text{Sol}$. Then, by (2.2), $\tau_a h(\mathfrak{r}) = h(g_a \mathfrak{r}) = h(x + a, y, z)$, and Lemma 2.7 tells us that $\tau_a h$ is in $\mathcal{H}^+(\mathcal{L}_a, \lambda)$. Now, by (1.5),

$$d(g_a \mathfrak{z}_n, \mathfrak{z}_n) = d_{\text{Sol}}((x_n + a, y_n, z_n), (x_n, y_n, z_n)) = d_{\mathbb{H}(\mathfrak{p})}((x_n + a, z_n), (x_n, z_n)).$$

Elementary properties of the hyperbolic metric imply that

$$d_{\mathbb{H}(\mathfrak{p})}((x_n + a, z_n), (x_n, z_n)) = d_{\mathbb{H}(\mathfrak{p})}((a, z_n), (0, z_n)) \leq d_{\mathbb{H}(\mathfrak{p})}((a, c), (0, c)) = d_a,$$

where $c = \inf_n z_n$. Let $C_{d_a}(\lambda)$ be the corresponding Harnack constant in Lemma 5.6. Then, using that $g(\cdot, \cdot | \lambda)$ is $\text{Sol}(\mathfrak{p}, \mathfrak{q})$ -invariant,

$$\mathbf{k}(g_a \mathfrak{r}, \mathfrak{z}_n | \lambda) = \frac{g(g_a \mathfrak{r}, \mathfrak{z}_n | \lambda)}{g(g_a \mathfrak{r}, g_a \mathfrak{z}_n | \lambda)} \frac{g(g_a \mathfrak{r}, g_a \mathfrak{z}_n | \lambda)}{g(0, \mathfrak{z}_n | \lambda)} \leq C_{d_a} \mathbf{k}(\mathfrak{r}, \mathfrak{z}_n | \lambda).$$

Letting $n \rightarrow \infty$, we obtain

$$\tau_a h(\mathfrak{r}) = h(g_a \mathfrak{r}) \leq C_{d_a} h(\mathfrak{r}) \quad \text{for all } \mathfrak{r} \in \text{Sol}.$$

Now minimality of h implies that the function $\tau_a h/h$ is constant. For $\mathfrak{r} = (x, y, z) \in \text{Sol}$,

$$d_{\text{Sol}}(g_a \mathfrak{r}, \mathfrak{r}) = d_{\mathbb{H}(\mathfrak{p})}((x + a, z), (x, z)) \rightarrow 0 \quad \text{if } z \rightarrow +\infty.$$

Therefore, the second statement in Lemma 5.6 implies that $h(g_a \mathfrak{r})/h(\mathfrak{r}) \rightarrow 1$ as $z \rightarrow +\infty$, and we conclude that $\tau_a h/h \equiv 1$. This proves statement (a) of the claim, and statement (b) follows by exchanging the roles of the x - and y -coordinates and changing the sign of z .

Now (z_n) must have a subsequence which converges to a limit in $[-\infty, +\infty]$. We may assume without loss of generality that (z_n) itself converges.

Case 1. $z_n \rightarrow \infty$. Then we can apply part (a) of the Claim, and conclude that h depends only on (y, z) . By Lemma 5.3, there is a function h_2 on $\mathbb{H}(\mathfrak{q})$ which is minimal in $\mathcal{H}^+(\mathcal{L}_{-a}^{\mathbb{H}(\mathfrak{q})}, \lambda)$, such that $h(x, y, z) = h_2(y, -z)$ for all $\mathfrak{r} = (x, y, z) \in \text{Sol}$.

Case 2. $z_n \rightarrow -\infty$. Then we can apply part (b) of the Claim, and again by Lemma 5.3, there is a function h_1 on $\mathbb{H}(\mathfrak{p})$ which is minimal in $\mathcal{H}^+(\mathcal{L}_a^{\mathbb{H}(\mathfrak{p})}, \lambda)$, such that $h(x, y, z) = h_1(x, z)$ for all $\mathfrak{r} = (x, y, z) \in \text{Sol}$.

Case 3. $z_n \rightarrow z_0 \in \mathbb{R}$. Then we can apply both parts (a) and (b) of the claim, and there is a function \tilde{h} on \mathbb{R} such that $h(x, y, z) = \tilde{h}(z)$ for all $\mathfrak{r} = (x, y, z) \in \text{Sol}$. It must be minimal both as a function $(x, z) \mapsto \tilde{h}(z)$ in $\mathcal{H}^+(\mathcal{L}_a^{\mathbb{H}(\mathfrak{p})}, \lambda)$ and as a function $(y, z) \mapsto \tilde{h}(-z)$ in $\mathcal{H}^+(\mathcal{L}_{-a}^{\mathbb{H}(\mathfrak{q})}, \lambda)$. Of course, it must also be a minimal element of $\mathcal{H}^+(\mathcal{L}_a^{\mathbb{R}}, \lambda)$. ■

Remark 6.2. If $h \in \mathcal{H}^+(\mathcal{L}_a^{\text{Sol}(p,q)}, \lambda)$ is minimal and depends only on z , then it must arise by lifting a minimal element of $h \in \mathcal{H}^+(\tilde{\mathcal{L}}_a, \lambda)$ from \mathbb{R} to Sol . That is, we must have $h(x, y, z) = e^{\alpha z}$, where $\alpha = \pm\sqrt{a^2 + 2\lambda} - a$. Furthermore, in this case, the function $(x, z) \mapsto e^{\alpha z}$ must be minimal in $\mathcal{H}^+(\mathcal{L}_a^{\mathbb{H}(p)}, \lambda)$, so that—by Lemma 5.1—we can only have the “+” sign, that is, $\alpha = \alpha(\lambda, a)$. We shall see below that the corresponding function can really be a minimal λ -eigenfunction on Sol only when $\lambda = \lambda_{\min}$. \square

Corollary 6.3. If $h \in \mathcal{H}^+(\mathcal{L}_a^{\text{Sol}(p,q)}, \lambda)$, where $\lambda \geq -a^2/2$, then there are nonnegative functions $h_1 \in \mathcal{H}^+(\mathcal{L}_a^{\mathbb{H}(p)}, \lambda)$ and $h_2 \in \mathcal{H}^+(\mathcal{L}_{-a}^{\mathbb{H}(q)}, \lambda)$ such that for all $\mathfrak{r} = (x, y, z) \in \text{Sol}(p, q)$,

$$h(x, y, z) = h_1(x, z) + h_2(y, -z). \quad \square$$

Proof. We see from Theorem 6.1 that the set of all minimal λ -eigenfunctions on $\text{Sol}(p, q)$ is contained in the union of the sets of minimal λ -eigenfunctions on $\mathbb{H}(p)$ and $\mathbb{H}(q)$, with a change of the sign of z for the latter, according to the above cases. Thus, taking into account Remark 6.2, $\mathcal{M}_{\min}(\lambda)$ can be parameterized by a subset of the disjoint union $\vartheta\mathbb{H}(p) \cup \vartheta^*\mathbb{H}(q) \cong (\vartheta\mathbb{H}(p) \times \{\varpi_q\}) \cup (\{\varpi_p\} \times \vartheta^*\mathbb{H}(q)) \subset \text{Sol}(p, q)$, or in other terms, of the “8”-shaped outer part of the geometric boundary of Sol (without the interiors of the two disks).

By Proposition 5.8, for every function $h \in \mathcal{H}^+(\mathcal{L}_a^{\text{Sol}(p,q)}, \lambda)$, there is a Borel measure $\nu = \nu^h$ on $\mathcal{M}_{\min}(\lambda)$ that yields the integral representation of h . Now let ν_1 be the restriction of ν to $\vartheta\mathbb{H}(p)$ and ν_2 the restriction to $\vartheta^*\mathbb{H}(q)$. Then we obtain, for every $\mathfrak{r} = (x, y, z) \in \text{Sol}$,

$$h(\mathfrak{r}) = \int_{\vartheta\mathbb{H}(p)} P_{p,a,\lambda}((x, z), \xi) d\nu_1(\xi) + \int_{\vartheta^*\mathbb{H}(q)} P_{q,-a,\lambda}((y, -z), \eta) d\nu_2(\eta) = h_1(x, z) + h_2(y, -z),$$

as proposed. \blacksquare

Corollary 6.4. The Laplacian $\mathcal{L}_a^{\text{Sol}(p,q)}$ has the (weak) Liouville property, that is, all bounded harmonic functions on Sol are constant, if and only if the rate of escape a vanishes. \square

Proof. If $a = 0$, then all bounded harmonic functions are constant by Corollary 6.3 and Remark 5.2. Conversely, if $a \neq 0$, then again by Remark 5.2, one of $\mathcal{L}_a^{\mathbb{H}(p)}$ and $\mathcal{L}_{-a}^{\mathbb{H}(q)}$ has nonconstant bounded harmonic functions, and they lift to harmonic functions on $\text{Sol}(p, q)$. \blacksquare

The last corollary, which was obtained in a very concrete, case-specific way, should be compared with the theorem of Karlsson and Ledrappier [28], which says that (under very general conditions) the weak Liouville property holds if and only if the rate of escape of Brownian motion is 0.

When $a \neq 0$, we have the following.

Corollary 6.5. (i) If $a > 0$, then every bounded harmonic function for $\mathfrak{L}_a^{\text{Sol}(p,q)}$ has the form $(x, y, z) \mapsto h_2(y, -z)$, where h_2 is a bounded harmonic function for $\mathfrak{L}_{-a}^{\mathbb{H}(q)}$.
(ii) If $a < 0$, then every bounded harmonic function for $\mathfrak{L}_a^{\text{Sol}(p,q)}$ has the form $(x, y, z) \mapsto h_1(x, z)$, where h_1 is a bounded harmonic function for $\mathfrak{L}_a^{\mathbb{H}(p)}$. \square

Proof. Let h be a bounded harmonic function on $\text{Sol}(p, q)$. We may assume without loss of generality that it is nonnegative. We decompose $h(x, y, z) = h_1(x, z) + h_2(y, -z)$ according to Corollary 6.3. Then both h_1 and h_2 are bounded harmonic. When $a > 0$, Remark 5.2 tells us that h_1 must be constant, so that we can “incorporate” it into h_2 . Analogously, when $a < 0$, the function h_2 must be constant. \blacksquare

Theorem 6.6. The minimal eigenfunctions in $\mathcal{H}^+(\mathfrak{L}_a^{\text{Sol}(p,q)}, \lambda)$, $\lambda \geq \lambda_{\min}$, are precisely the functions

$$(x, y, z) \mapsto P_{p,a,\lambda}((x, z), \xi) \quad \text{and} \quad (x, y, z) \mapsto P_{q,-a,\lambda}((y, -z), \eta), \quad \xi, \eta \in \mathbb{R},$$

and in addition, when $\lambda = \lambda_{\min}$, the function

$$(x, y, z) \mapsto e^{-az}. \quad \square$$

Proof. Combining Theorem 6.1 with Lemma 5.1, we see that each minimal λ -eigenfunction on Sol must be of the form

$$(x, y, z) \mapsto P_{p,a,\lambda}((x, z), \xi) \quad \text{where } \xi \in \partial^*\mathbb{H}(p), \quad \text{or}$$

$$(x, y, z) \mapsto P_{q,-a,\lambda}((y, -z), \eta) \quad \text{where } \eta \in \partial^*\mathbb{H}(q).$$

We have to show that for $\xi \neq \varpi_p$ and for $\eta \neq \varpi_q$, the respective functions are indeed all minimal. Furthermore, we have to show that for $\xi = \varpi_p$ and for $\eta = \varpi_q$, the two resulting functions are *not* minimal λ -eigenfunctions on Sol , unless $\lambda = \lambda_{\min}$. In this last case both coincide and are equal to e^{-az} .

So first we show minimality of $(x, y, z) \mapsto P_{p,a,\lambda}((x, z), \xi)$ with $\xi \in \partial^*\mathbb{H}(p)$. Suppose that $P_{p,a,\lambda}((x, z), \xi) \geq h(x, y, z)$ for all $\mathfrak{z} = (x, y, z)$, where $h \in \mathcal{H}^+(\mathfrak{L}_a^{\text{Sol}(p,q)}, \lambda)$.

We decompose $h(x, y, z) = h_1(x, z) + h_2(y, -z)$ according to Corollary 6.3. By minimality of $P_{p,a,\lambda}((\cdot, \cdot), \xi)$ in $\mathcal{H}^+(\mathcal{L}_a^{\mathbb{H}(p)}, \lambda)$ (Lemma 5.1), we must have $h_1 = c \cdot P_{p,a,\lambda}((\cdot, \cdot), \xi)$, where $0 \leq c \leq 1$. If $c = 1$, we are done. If $c < 1$, then we get

$$P_{p,a,\lambda}((x, z), \xi) \geq \frac{1}{1-c} h_2(y, -z) = \int_{\vartheta \mathbb{H}(q)} P_{q,-a,\lambda}((y, -z), \eta) d\nu(\eta) \quad \text{for all } (x, y, z) \in \text{Sol},$$

where ν is a Borel measure on $\vartheta \mathbb{H}(q)$. Setting $y = z = 0$, we obtain

$$P_{p,a,\lambda}((x, 0), \xi) \geq \nu(\vartheta \mathbb{H}(q)) \quad \text{for all } x \in \mathbb{R}.$$

If $x \rightarrow \infty$, then we see from the formula for $P_{q,-a,\lambda}$ of Lemma 5.1 that the left-hand side in the last inequality tends to 0. Therefore, $\nu(\vartheta \mathbb{H}(q)) = 0$, whence $h_2 \equiv 0$, contradicting the assumption that $c < 1$.

The proof of minimality of $(x, y, z) \mapsto P_{q,-a,\lambda}((x, -z), \eta)$, where $\eta \in \vartheta^* \mathbb{H}(q)$, follows as usual by exchanging the roles of the x - and y -variables.

Next, let $\xi = \varpi_p$ and $\lambda > \lambda_{\min}$, so that we are considering the function

$$(x, y, z) \mapsto P_{p,a,\lambda}((x, z), \varpi_p) = e^{\alpha(\lambda,a)z}.$$

If it were minimal in $\mathcal{H}^+(\mathcal{L}_a^{\text{Sol}(p,q)}, \lambda)$, then by Lemma 5.3, also the function $(y, z) \mapsto e^{-\alpha(\lambda,a)z}$ would have to be minimal in $\mathcal{H}^+(\mathcal{L}_{-a}^{\mathbb{H}(q)}, \lambda)$, which is not the case by Lemma 5.1.

Analogously, when $\lambda > \lambda_{\min}$, the function

$$(x, y, z) \mapsto P_{q,-a,\lambda}((y, -z), \varpi_q) e^{-\alpha(\lambda,-a)z}$$

cannot be minimal in $\mathcal{H}^+(\mathcal{L}_a^{\text{Sol}(p,q)}, \lambda)$.

Finally, consider the case $\lambda = \lambda_{\min}$ and the function $(x, y, z) \mapsto e^{-az}$ in $\mathcal{H}^+(\mathcal{L}_a^{\text{Sol}(p,q)}, \lambda)$. We use a well-known trick, conjugating our operator with this exponential: suppose that $e^{-az} \geq h(x, y, z)$, where $h \in \mathcal{H}^+(\mathcal{L}_a^{\text{Sol}(p,q)}, \lambda)$. Then a straightforward computation shows that the function $\tilde{h}(x, y, z) = e^{az} h(x, y, z)$ is in $\mathcal{H}^+(\mathcal{L}_0^{\text{Sol}(p,q)}, 0)$, that is, it is bounded harmonic, and the new rate of escape is 0. By Corollary 6.4, \tilde{h} is constant. This proves minimality of $(x, y, z) \mapsto e^{-az}$ in $\mathcal{H}^+(\mathcal{L}_a^{\text{Sol}(p,q)}, \lambda)$. ■

Our results tell us that the *Poisson boundary* of Brownian motion with drift on Sol is the “outer” boundary

$$(\vartheta^* \mathbb{H}(p) \times \{\varpi_q\}) \cup (\{\varpi_p\} \times \vartheta^* \mathbb{H}(q)) \cup \{(\varpi_p, \varpi_q)\}$$

together with the limit distribution provided by Proposition 4.2. Indeed, for $a < 0$, it is just the first of these three pieces, because the limit distribution is supported by that piece. For $a > 0$, it is just the second piece, and for $a = 0$, it is trivial, that is, the singleton of the third piece. Here, we do not go into details regarding the construction of the Poisson boundary. (In short and informally, it is the smallest probability space that gives rise to an integral representation of all bounded harmonic functions and at the same time provides a model for the limit behavior of the process at infinity.) The reader is referred to the body of work of Kaimanovich, for example, [27].

Regarding the Martin boundary (which is a metric space, while the Poisson boundary is a measure space), our results underline the evidence that $\mathcal{M}(\lambda_{\min})$ is the boundary in the geometric compactification that we have described in Section 5, while for $\lambda > \lambda_{\min}$ it should be bigger: one first should consider the *horocyclic compactification* of $\mathbb{H}(\mathfrak{p})$, which can be built from the usual one as follows. Replace the boundary point $\varpi_{\mathfrak{p}}$ by the set $\{\varpi_{\mathfrak{p}}^{\zeta} : \zeta \in [-\infty, \infty]\}$, which carries the topology of the extended real line. Furthermore, modify the topology by saying that in the new compactification, $(x, z) \rightarrow \varpi_{\mathfrak{p}}^{\zeta}$ if $|x| \rightarrow \infty$ and $z \rightarrow \zeta$. Then we expect that the Martin compactification of $\text{Sol}(\mathfrak{p}, \mathfrak{q})$ for $\lambda > \lambda_{\min}$ is the closure of Sol in the direct product of the horocyclic compactifications of the two hyperbolic planes. This evidence comes from the strong analogy with the DL-graphs (the horocyclic product of two homogeneous trees), see [11]; the rigorous proof still has to be carried out.

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