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RENEWAL THEORY ON THE AFFINE GROUP OF AN ORIENTED TREE

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ABSTRACT. The affine group of a tree is the group of the isometries of a homogeneous tree that fix an end of its boundary. Consider a probability measure μ on this group and the associated random walk. The main goal of this paper is to determine the accumulation points of the potential kernel

$$g * U = g * \sum_{n=0}^{\infty} \mu^{(n)}$$

when g tends to infinity. In particular we show that under suitable regularity hypotheses this kernel can be continuously extended to the tree's boundary and we determine the limit measures.

Key words: Random walk, renewal theory, affine group, tree, p-adic rationals

CONTENTS

Introduction 1
1. Random walks on the affine group of a tree 3
1.1. Oriented tree 3
1.2. The affine group of the tree 4
1.3. Non-exceptional subgroup 5
1.4. Random walks on $\mathrm{Aff}(\mathbb{T})$ 6
1.5. Remarkable examples 6
1.6. Renewal on the tree 9
2. Preliminary results and a renewal equality 10
2.1. Convergence of the random walk to the boundary 10
2.2. Action on $\partial^*\mathbb{T}$ 12
2.3. A renewal equality 15
2.4. The action on $\mathrm{Hor}(\Gamma)$ and a renewal equality on the group 17
3. Limit measures of the potential kernel 18
3.1. The limits of the potential kernel at $\partial^*\mathbb{T}$ 19
3.2. The limit of the potential kernel at ω 25
References 27

INTRODUCTION

Consider a transient random walk with law μ on a locally compact group. Its potential measure $U = \sum_{n=0}^{\infty} \mu^{(n)}$ is a Radon measure and its (right) potential kernel $g * U$ is, when g varies on the group, a family of measures that is vaguely relatively compact. Renewal theory consists in studying the limit behavior of this family when g goes to infinity, determining the limit measures and the geometrical

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directions along which it converges. On Abelian groups, this problem has been completely solved (cf. [13]): there are not more than two accumulation points (the null measure and the Haar measure) and there is a non-zero limit if and only if the group is a compact extension of $\mathbb{Z}$ or $\mathbb{R}$. The work of L.Elie on the affine group of the real line and on almost connected Lie groups ([6]) has shown that for non-unimodular groups we may have a quite different behavior. Namely there exists an infinite number of limit measures, and the Haar measure cannot be among them.

In this paper we leave the Euclidean setting to deal with this kind of question on the group of affine transformations of the homogeneous tree, $\mathrm{Aff}(\mathbb{T})$, that is the group of tree isometries that fix an end of the boundary. D.Cartwright, V.Kaimanovich and W.Woess have given in [4] a first detailed study of random walks on this group and we refer to this article for a comprehensive introduction.

The affine group of the tree contains the affine group of the p-adic numbers, $\mathrm{Aff}(\mathbb{Q}_p)$ (or more generally of a local field), that is the group of matrices of the form $\left[\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right]$ where the coefficients $a \neq 0$ and b are p-adic valued. The tree is in fact the Bruhat-Tits building of the invertible 2×2 matrices on $\mathbb{Q}_p$ and its affine group acts on the tree analogously as the real affine group, $\mathrm{Aff}(\mathbb{R})$, acts on the hyperbolic plane $\mathbb{H}^2$, that is by isometries and fixing a boundary point. On the other hand, the structural analogies apart, the real affine group and the affine group of the tree present remarkable differences. While $\mathrm{Aff}(\mathbb{R})$ can be identified with the group of all isometries that fix a boundary point, the affine group of the tree is significantly bigger and more complex than $\mathrm{Aff}(\mathbb{Q}_p)$ and it contains other interesting subgroups such as the lamplighter group or automatic groups. This complexity is mainly due to the fact that the graph structure of the tree is much less rigid than the hyperbolic plane, in the sense that the local behavior of an isometry does not determine how it acts globally.

The main goal of this paper is to show that the potential kernel of a random walk supported by anon-exceptional subgroup of $\mathrm{Aff}(\mathbb{T})$ can be continuously extended to the boundary of the tree, and to give a description of the limit measures by mean of the invariant measure on the boundary and the counting measure on the integers (Theorems 3.6 and 3.7). These last conclusions are particularly interesting in view of further studies, namely for the characterization of the Martin boundary points and thus in the representation of the invariant measures.

Our results are obtained partially by adapting Elie's methods that involve the characterization of the periods of the limit measures, partially by using a weighted renewal equality (Corollary 2.8) whose analogue over $\mathbb{R}$ is due to M.Babillot, Ph.Bougerol and L.Elie [1]. In a general setting we require, besides weak moment conditions, that the the step law of the random walk is spread out. However, for random walks supported by groups that act on the tree in a sufficiently homogeneous way (such as $\mathrm{Aff}(\mathbb{Q}_p)$) we are able to avoid this last continuity hypothesis for the limit towards all boundary points except for the one that is fixed by the group.

The paper is structured as follows:

In Section 1, we introduce the structures we are working on (the oriented tree, the affine group and its non-exceptional subgroups) and the probabilistic objects we are going to study (random walks and potential kernel).

In Section 2, we give some preliminary results concerning the convergence and the action of the random walks on the tree's boundary and obtain a measure equality for the potential kernel on the group.

In Section 3, we prove our main results. We start by determining some invariance properties of the limit measures and then we characterize the limits of the potential kernel.

1. Random walks on the affine group of a tree

1.1. Oriented tree. We consider the *homogeneous tree* $\mathbb{T}$ of degree $q+1$, i.e. the connected non-oriented graph without cycles whose vertices have exactly $q+1$ neighbors, equipped with the usual graph distance

$$d(x,y) = \text{number of edges between } x \text{ and } y.$$

The set of infinite geodesic rays that start from some vertex and go to infinity, quotiented by the equivalence relation that identifies two geodesics when they coincide but for a finite number of vertices, give the geometrical *boundary* of the tree, $\partial\mathbb{T}$. The union $\mathbb{T} \cup \partial\mathbb{T}$, equipped with the topology of the infinite cones starting from a vertex, is then a compact set where $\mathbb{T}$ is a dense open sub-set.

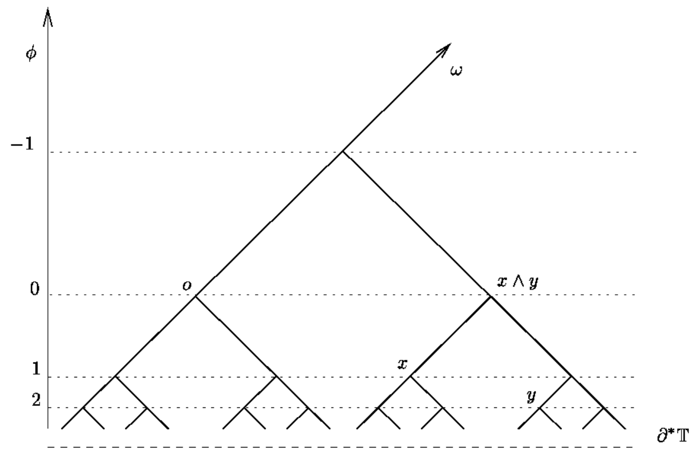
A partial order of the tree is given, fixing an end ω in $\partial\mathbb{T}$ and setting for all $x \neq y$ in $\mathbb{T} \cup \partial\mathbb{T}$

$$x \wedge y = \text{ first common vertex of } \overline{x\omega} \text{ and } \overline{y\omega},$$

where $\overline{x\omega}$ is the geodesic starting at x and in the class of ω, and $x \wedge x = x$. We write

$$x \succeq y \quad \Leftrightarrow \quad x = x \wedge y.$$

One can imagine the oriented tree as an infinite genealogical tree, where ω represents the *mythical ancestor*, every vertex has q sons and a father and $x \succeq y$ if and only if y is a descendent of x.



Let us fix a reference vertex o in $\mathbb{T}$ called *origin*. The height function ϕ from $\mathbb{T}$ to $\mathbb{Z}$ is

$$\phi(x) := d(x, x \wedge o) - d(o, x \wedge o),$$

also known as the *Busemann* function, and represents the generation number of x.

Let us consider the bottom boundary of the tree

$$\partial^*\mathbb{T} = \partial\mathbb{T} - \{\omega\}.$$

The function ϕ induces an ultra-metric distance on $\mathbb{T} \cup \partial^*\mathbb{T}$ defined by

$$\Theta(\alpha,\beta) := \begin{cases} q^{-\phi(\alpha\wedge\beta)} & \text{if } \alpha \neq \beta \in \mathbb{T} \cup \partial^*\mathbb{T} \\ 0 & \text{if } \alpha = \beta \end{cases}.$$

1.2. **The affine group of the tree.** The group of isometries of the tree $(\mathbb{T}, d)$ has a natural continuous action on the boundary, obtained by the action on the geodesics.

The *affine group of the tree* is the subgroup of the isometries that fix the end ω

$$\mathrm{Aff}(\mathbb{T}) := \{g \in \mathrm{Iso}(\mathbb{T}) : g\omega = \omega\},$$

that is the subgroup that preserves the order induced by ω,

$$g(x \wedge y) = gx \wedge gy \qquad \text{for all } g \in \mathrm{Aff}(\mathbb{T}).$$

The group $\mathrm{Aff}(\mathbb{T})$ is equipped with the topology of pointwise convergence on the tree, where a base of open neighborhoods of an affinity γ is given by the sets

$$V(\mathbf{x} \to \mathbf{y}) := \{g \in \mathrm{Aff}(\mathbb{T}) : g\mathbf{x} = \mathbf{y}\}$$

for every finite set of vertices $\mathbf{x} = \{x_1, \dots, x_n\}$ and with $\mathbf{y} = \gamma\mathbf{x} = \{\gamma x_1, \dots, \gamma x_n\}$. These sets are simultaneously open and compact, therefore $\mathrm{Aff}(\mathbb{T})$ is a locally compact totally disconnected group.

The semi-norm

$$|g| = d(go, o)$$

on $\mathrm{Aff}(\mathbb{T})$ is symmetric, $|g| = \left|g^{-1}\right|$, and verifies to $|g_1 g_2| \leq |g_1| + |g_2|$. The set of the affinities of zero norm is $V(o \to o)$, a compact subgroup.

1.2.1. *Drift of an affinity and the horocyclic group.* As the affinities respect the order and the distance on the tree, for every couple of vertices x and y one has

$$\phi(gx) - \phi(gy) = \phi(x) - \phi(y).$$

The homomorphism:

$$\begin{array}{rcl} \phi : \mathrm{Aff}(\mathbb{T}) & \to & \mathbb{Z} \\ g & \mapsto & \phi(gx) - \phi(x) = \phi(go), \end{array}$$

does not depend on the choice of the point x and contains the information on the vertical action of an affinity on the tree. It also indicates whether the action of g on the bottom boundary $\partial^*\mathbb{T}$ dilates or contracts, in fact for every couple of ends α and β in $\partial^*\mathbb{T}$

$$\Theta(g\alpha, g\beta) = q^{-\phi(g\alpha \wedge g\beta)} = q^{-\phi(g)}\Theta(\alpha, \beta). \tag{1.1}$$

The *horocyclic group* of the tree is the subgroup of the affine group that fixes the heights

$$\mathrm{Hor}(\mathbb{T}) := \ker \phi = \{g \in \mathrm{Aff}(\mathbb{T}) : \phi(gx) = \phi(x) \quad \forall x \in \mathbb{T}\}.$$

It follows from (1.1) that $\mathrm{Hor}(\mathbb{T})$ is the group of all isometries of $(\partial^*\mathbb{T}, \Theta)$.

Instead of working on the whole affine group, we will be often interested in some closed subgroup Γ of $\mathrm{Aff}(\mathbb{T})$. In this case we consider ϕ as an homomorphism from Γ to $\mathbb{Z}$ and set

$$\mathrm{Hor}(\Gamma) := \ker \phi = \mathrm{Hor}(\mathbb{T}) \cap \Gamma.$$

1.2.2. *Algebraic structure of* $\mathrm{Aff}(\mathbb{T})$. For the sake of simplicity we always suppose that the homomorphism ϕ from Γ to $\mathbb{Z}$ is surjective. Then for all $s \in \Gamma$ such that $\phi(s) = 1$, every $g \in \Gamma$ has a unique decomposition as a product of an element of the horocyclic group and a power of s

$$g = b(g)s^{\phi(g)} \quad \text{where} \quad b(g) := gs^{-\phi(g)} \in \mathrm{Hor}(\Gamma). \tag{1.2}$$

Thus, if we identify $\mathbb{Z}$ with the subgroup generated by s, the group Γ is the semi-direct product

$$\begin{array}{rcl} \mathrm{Hor}(\Gamma) \rtimes_s \mathbb{Z} & \cong & \Gamma \\ (b, h) & \mapsto & bs^h. \end{array}$$

Note that the decomposition of $\mathrm{Aff}(\mathbb{T})$ as semi-direct product of $\mathbb{Z}$ and $\mathrm{Hor}(\mathbb{T})$ depends on the choice of the element s, and we call it *reference homothety.* We denote by $\alpha = \alpha_s$ the unique end of $\partial^*\mathbb{T}$ fixed by s (for its existence see for instance [17]). The homothety s acts by translation on the geodesic $\overline{\alpha\omega}$, that may then be considered as a "main branch" of the tree. To choose a reference homothety is equivalent to select a center α of the bottom boundary and a canonical identification between the sub-trees that branch from $\overline{\alpha\omega}$.

1.2.3. *Rotations.* In some sense the horocyclic group, that is the group of all isometries of the bottom boundary, plays the role of the group of translations in the real case; but in our case the action on $\partial^*\mathbb{T}$ is not simple. In fact the stabilizer of an end $\alpha \in \partial^*\mathbb{T}$ in the horocyclic group , that is the group of *rotations* of center α, is not trivial and is the compact subgroup

$$K_\alpha = K_\alpha(\Gamma) = \{r \in \mathrm{Hor}(\Gamma) : \, r\alpha = \alpha\}\,.$$

It is worthwhile observing that, contrary to what happens on Lie groups, the identification induced by the reference homothety s is completely arbitrary, because the structure of the tree is much less rigid than in the analogous continuous spaces. One of the first consequence is that a rotation does not commute with a homothety of the same center; in fact whenever a rotation r acts on two sub-trees in different ways (according to the identification induced by s), one has $sr \neq rs$. Moreover as there is no rigidity, we do not have a finite set of points whose images uniquely determine a rotation or an affinity, but given any compact set C in $\mathbb{T}$ (or in $\partial^*\mathbb{T}$) one may find two affinities that act in the same way on C and are different on its complement.

1.2.4. *Compactification and group boundary.* The action on the tree enables us to give a natural compactification of any closed subgroup of $\mathrm{Aff}(\mathbb{T})$. In fact it is easy to see that whenever for a sequence $\{g_n\}_n$ in $\mathrm{Aff}(\mathbb{T})$ there exists a vertex $x \in \mathbb{T}$ such that $\{g_n x\}_n$ converges to an end β in $\partial\mathbb{T}$, then for all $y \in \mathbb{T}$ also $\{g_n y\}_n$ converges to β. We then say that the sequence $\{g_n\}_n$ *converges to* β and we compactify $\mathrm{Aff}(\mathbb{T})$ in $\mathrm{Aff}(\mathbb{T}) \cup \partial\mathbb{T}$ setting

$$g_n \to \beta \in \partial\mathbb{T} \Leftrightarrow \exists\,(\text{or } \forall)\, x \in \mathbb{T} \,:\, g_n x \to \beta.$$

The boundary of a subgroup Γ of $\mathrm{Aff}(\mathbb{T})$ is then the set of the accumulation points of Γ in $\partial\mathbb{T}$ and is denoted by $\partial\Gamma$.

1.3. **Non-exceptional subgroup.** We focus our study on random walks supported on subgroups of $\mathrm{Aff}(\mathbb{T})$ that are non-degenerate. More precisely we deal with closed subgroups, Γ, that are *non-exceptional*, i.e. satisfy one of the following equivalent conditions (cf. [4]):

- Γ is not contained in $\mathrm{Hor}(\mathbb{T})$ and it does not fix any end in $\partial^*\mathbb{T}$
- Γ is non-unimodular
- $\partial\Gamma$ is infinite

In the usual parallelism with the real affine group, this is equivalent to asking that the group Γ is neither a group of translations nor a group of rotations and homotheties. Another important property of non-exceptional subgroups is that all their orbits are dense in the bottom boundary of the group

$$\partial^*\Gamma = \partial\Gamma - \{\omega\}\,.$$

By (1.2), this holds also for $\mathrm{Hor}(\Gamma)$. When the group is also closed then its action on $\partial^*\Gamma$ is transitive, i.e. $\Gamma\beta = \partial^*\Gamma$ for all $\beta \in \partial^*\Gamma$. Indeed, for every $\beta \in \partial^*\Gamma$ consider a sequence b_n in $\mathrm{Hor}(\Gamma)$ such that $b_n\alpha$ converge to β. This is a relatively compact sequence and every accumulation point b (which is in $\mathrm{Hor}(\Gamma)$, since it is closed) is such that $b\alpha = \beta$.

Let $s \in \Gamma$ be a reference homothety of center α. As $K_\alpha = K_\alpha(\Gamma)$ is the stabilizer of the end $\alpha \in \partial^*\Gamma$ in $\mathrm{Hor}(\Gamma)$, the subgroup $K_\alpha \rtimes_s \mathbb{Z}$ is the stabilizer of α in Γ; thus we have the following identifications by homeomorphisms:

$$\Gamma / (K_\alpha \rtimes_s \mathbb{Z}) = \mathrm{Hor}(\Gamma)/K_\alpha = \partial^*\Gamma.$$

1.4. **Random walks on** $\mathrm{Aff}(\mathbb{T})$**.** Let μ be a probability measure on $\mathrm{Aff}(\mathbb{T})$ and $\{X_n\}_{n\in\mathbb{N}}$ a sequence of random variables defined on the probability space $(\Omega, \mathbb{P})$ and with values in $\mathrm{Aff}(\mathbb{T})$, independent and identically distributed with law μ. The *left and right random walks* are the Markov chains on $\mathrm{Aff}(\mathbb{T})$ defined by iterated products of the X_n on the left and on the right, respectively,

$$L_n = X_n X_{n-1} \cdots X_1 \qquad \text{and} \qquad R_n = X_1 \cdots X_{n-1} X_n$$

and $L_0 = R_0$ are equal to the identity e.

Although in terms of associated trajectory spaces, these two processes are different, for every fixed time n have the same law, the n-th convolution power of μ

$$L_n \overset{\text{law}}{=} R_n \sim \mu^{(n)}.$$

1.4.1. *Hypotheses.* All our results concern random walks whose action on the tree is sufficiently complete; namely, we shall always assume the following *non-degeneracy hypotheses:* the closed subgroup generated by the support of μ

$$\Gamma := \overline{< \mathrm{supp}\mu >}$$

is non-exceptional and, for sake of simplicity, that $\phi(\Gamma) = \mathbb{Z}$.

We also need some moment hypothesis that may vary according to the type of results that we want to obtain. We always suppose that the projection of the random walk on $\mathbb{Z}$ is integrable,

$$\mathbb{E}[|\phi(X_1)|] < +\infty.$$

Most of the times we also require a *moment of first order* for the random walk on the group

$$\mathbb{E}[|X_1|] < +\infty.$$

When projection of the random walk on $\mathbb{Z}$ is recurrent, we shall need a *moment of order* $2 + \epsilon$, namely

$$\mathbb{E}\Big[\phi(X_1)^2 + |b(X_1)|^{2+\varepsilon}\Big] < +\infty$$

for some $\varepsilon > 0$.

For generic random walks on $\mathrm{Aff}(\mathbb{T})$ we also require a continuity condition, namely that the measure μ is *spread out* (i.e. there exists a convolution power $\mu^{(n)}$ that is non-singular with respect to the Haar measure of Γ).

1.4.2. *Drift of the random walk.* A crucial role in the study of the random walks R_n and L_n is played by their projection on $\mathbb{Z}$, that is by the random walk

$$S_n = \phi(X_1) + \cdots + \phi(X_n) = \phi(L_n) = \phi(R_n).$$

Its mean is called *drift of* μ

$$\mu(\phi) = \mathbb{E}[\phi(X_1)]$$

and it is the parameter that enables to classify the different types of behavior.

1.5. **Remarkable examples.** Since the tree has a very lax structure, the affine group of the tree is very complex and there are many different ways to construct random walks supported by non-exceptional subgroups. We present here some remarkable examples.

1.5.1. *Random walks on the tree.* The simplest example is given by Markov chains on the tree that are invariant under the transitive action of a subgroup Γ of $\mathrm{Aff}(\mathbb{T})$, as for instance the nearest neighbor random walk on the tree where from a vertex one goes to the father with probability α and to every son with probability $(1-\alpha)/q$. This type of process can be obtained from a random walk on Γ, whose law is invariant by the right action of the stabilizer $V(x)$ of a vertex $x \in \mathbb{T}$. In that case the process $Z_n = R_n V(x)$ is a Markov chain on $\Gamma/V(x) = \mathbb{T}$ starting in x and homogeneous under the action of Γ.

One can show that, since the stabilizer $V(x)$ is an open and compact subgroup, every measure that is right invariant by the action of $V(x)$ has a continuous density with respect to Haar measure. Thus every Markov chain on the tree that is invariant under the transitive action of an non-exceptional subgroup may be considered a random walk on the group whose law is spread out, and therefore all our results translated to this setting.

1.5.2. *p-adic affine group.* One of the most interesting non-exceptional subgroups of the affine group of the tree is the affine group of rational p-adics. We will often refer to it because, apart for its intrinsic interest, it is the more natural analogue of the real affine group and thus it allows to stress the similarities but also the main differences between the real and tree settings.

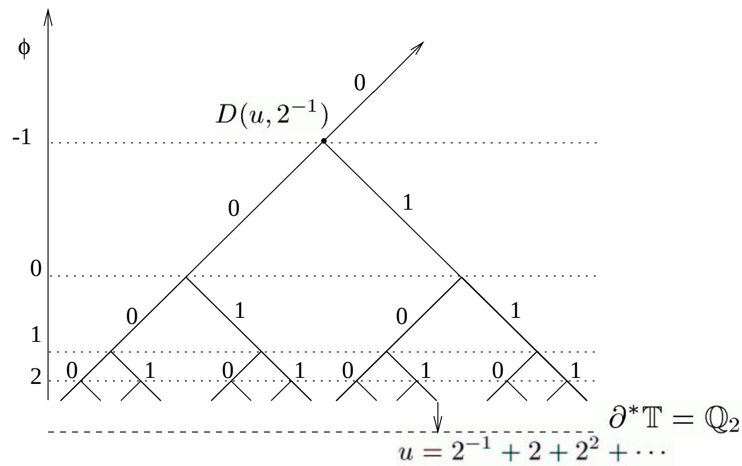
Let p be a prime number. Consider the integer evaluation v_p on the rational numbers that measures how much a number is divisible by p, i.e. we set

$$v_p(u) = \max\{k \in \mathbb{Z} : p^{-k}u \in \mathbb{Z}\} \quad \text{for } u \in \mathbb{N}$$

and $v_p(\frac{u}{w}) = v_p(u) - v_p(w)$ for $\frac{u}{w} \in \mathbb{Q}^*$. The field of rational p-adic rationals is then the completion of $\mathbb{Q}$ equipped with the ultra-metric norm

$$|u|_p = p^{-v_p(u)} \quad \text{for all } u \in \mathbb{Q}.$$

There exists a strict relationship between the p-adic rational and the oriented tree of degree $p+1$ (cf. Serre [15]), since it is possible to consider the tree as the set of the discs in $\mathbb{Q}_p$. First observe that, because the evaluation v_p is integer valued and $|\cdot|_p$ has the ultra-metric property, the set of all discs of $\mathbb{Q}_p$ is countable and, if it is equipped with the natural order given by inclusion, it has the structure of an oriented tree: each disc of radius p^k contains exactly p discs of radius p^{k-1}, its sons, and the disc $D(u,p^k)$ of center u and radius p^k has the disc $D(u,p^{k+1})$ as father.



If $D(0,1)$ is the origin of the tree, then the Busemann function is

$$\phi(D(u,p^k)) = -k.$$

One has a one to one mapping (that is in fact an isometry) between $\partial^*\mathbb{T}$ and $\mathbb{Q}_p$, associating the decreasing sequence $\{D(u,q^{-k})\}_{k\in\mathbb{N}}$ with the element $u = \bigcap_{k\in\mathbb{N}} D(u,q^{-k})$ of $\mathbb{Q}_p$.

Like in the real case, the p-adic affine group, $\text{Aff}(\mathbb{Q}_p)$, is the set of the mappings of the form

$$g = (t,a) : u \mapsto au + t \quad \text{with } a \in \mathbb{Q}_p^* \text{ and } t \in \mathbb{Q}_p.$$

and it can be realized as the group of matrices

$$\text{Aff}(\mathbb{Q}_p) = \left\{ \begin{bmatrix} a & t \\ 0 & 1 \end{bmatrix} : a \in \mathbb{Q}_p^* \text{ and } t \in \mathbb{Q}_p \right\}.$$

As affinities send discs on discs and respect the inclusion order, they constitute a subgroup of $\text{Aff}(\mathbb{T})$ that is closed and non-exceptional.

Since every affinity (t,a) is the composition of a translation $u \mapsto u+t$ and of a roto-homothety $u \mapsto au$, it is natural to see $\text{Aff}(\mathbb{Q}_p)$ as a semi-direct product $\mathbb{Q}_p \rtimes \mathbb{Q}_p^*$. However this decomposition does not coincide with the one we have introduced earlier as semi-direct product of $\text{Hor}(\mathbb{Q}_p)$ and of $\mathbb{Z}$. In fact as

$$\phi((t,a)) = v_p(a),$$

the horocyclic group is

$$\text{Hor}(\mathbb{Q}_p) = \left\{ \begin{bmatrix} a & t \\ 0 & 1 \end{bmatrix} \in \text{Aff}(\mathbb{Q}_p) : |a| = 1 \right\} = \mathbb{Q}_p \rtimes \mathbb{Z}_p$$

where $\mathbb{Z}_p$ is the ring of p-adic integers, that coincide with the group of rotation of center 0, while $\mathbb{Q}_p$ can be identified with the group of translations.

The p-adic affine group has much more similarities with the real case than a generic subgroup of $\text{Aff}(\mathbb{T})$. First of all in the p-adic setting the tree has an algebraic structure that is much more stiff that the one of a simple graph: every affinity is characterized by two parameters (t and a), and it is therefore uniquely determined when one knows how it acts on two points of $\mathbb{Q}_p = \partial^*\mathbb{T}$. Secondly, but not less important, the group of roto-homotheties of center 0, i.e. its stabilizer

$$K_0(\mathbb{Q}_p) \times \mathbb{Z} = \mathbb{Q}_p^*$$

is Abelian. Finally the bottom boundary $\partial^*\mathbb{T}$ is identified with $\mathbb{Q}_p$ and it has then the structure of an Abelian group. These properties permit us to obtain stronger results for random walks on the p-adic affine group than in the general setting.

One of the interests in the p-adic affine group is linked with the study of random walk on the group of affine transformations whose coefficients can take only rational values, $\text{Aff}(\mathbb{Q})$. This group may be naturally be regarded as a dense subgroup of the real or of the p-adic affine group, according to the metric one considers. As it has already been pointed out in previous works (see Kaimanovich [10]) from a measure theoretic point of view the behavior of random walks on $\text{Aff}(\mathbb{Q})$ is not necessarily related to the Euclidean metric, and a complete understanding may be obtained by a simultaneous immersion in $\text{Aff}(\mathbb{R})$ and in all the groups $\text{Aff}(\mathbb{Q}_p)$ where p is prime.

1.5.3. *Lamplighter group.* Another algebraic structure on the tree, different from the p-adic one, but that guarantees the same regularity properties, is given identifying the tree with sequences of integer numbers modulo q, i.e $\mathbb{Z}/q\mathbb{Z}$, and considering the action of the Lamplighter group, i.e. of the wreath product $(\mathbb{Z}/q\mathbb{Z}) \wr \mathbb{Z}$. More precisely let

$$\mathcal{Z}_q = \{\sigma : \mathbb{Z} \to \mathbb{Z}/q\mathbb{Z} \,|\, \sigma \,\text{has finite support}\}$$

and, for every $k \in \mathbb{Z}$, consider the equivalence relations such that if σ_k is the class of $\sigma \in \mathcal{Z}_q$ then $\sigma_k = \tau_k$ if and only if $\sigma(n) = \tau(n)$ for all $n \leq k$. One can then identify the oriented tree of degree $q+1$ with the set $\{\sigma_k \,|\, \sigma \in \mathcal{Z}_q \text{ and } k \in \mathbb{Z}\}$ in such a way that σ_k is the father of σ_{k+1}. The Lamplighter group, that can also be seen as the semi-direct product $\mathcal{Z}_q \rtimes \mathbb{Z}$, acts on the tree by the usual sum of $\mathcal{Z}_q$ and by shift and it is then a non-exceptional (but non-closed) subgroup of $\mathrm{Aff}(\mathbb{T})$.

1.6. **Renewal on the tree.** As we assumed that the group Γ generated by the support of μ is non-exceptional, thus non-unimodular, a fundamental result (cf. [9]) ensures that our random walks are always transient, i.e. almost surely they visit every compact set only a finite number of times. Their *potential measure,* i.e.

$$U(A) = \sum_{n=0}^{\infty} \mu^{(n)}(A) = \mathbb{E}\left[\sum_{n=0}^{\infty} 1_{[L_n \in A]}\right] = \mathbb{E}\left[\sum_{n=0}^{\infty} 1_{[R_n \in A]}\right],$$

is then a Radon measure on Γ. The *(right) potential kernel* is the family of measures

$$g * U(A) = \int_{\Gamma} 1_A(gx)\, U(dx) = \mathbb{E}\left[\sum_{n=0}^{\infty} 1_{[gR_n \in A]}\right]$$

where $g \in \Gamma$. This is the expected number of visits in the set $A \subset \Gamma$ for the right random walk starting in g. By the maximum principle, this family is bounded for every compact set A when g varies in Γ and thus is vaguely relatively compact. Its limit measures when g goes to infinity are Radon measures on the group Γ that are right μ-excessive, this is, they satisfies the inequality $\nu * \mu \leq \nu$ (they will turn out to be μ-invariant).

The goal of this paper is to study these measures, to describe their properties and to determine the directions of convergence of the potential kernel. In section 3, we will state and prove in detail our main results, which may be summarized in the following

Theorem. *Suppose that the measure μ is spread out and that satisfies suitable moment conditions. Then the potential kernel $g * U$ can be continuously extended to $\partial\Gamma$ .Furthermore*

$$\lim_{g \to \omega} g * U = 0$$

and for all $\beta \in \partial^*\mathbb{T}$, if $s \in \Gamma$ is a reference homothety of center α and $b \in \Gamma$ is such that $b\alpha = \beta$

$$\lim_{g \to \beta} g * U = \nu_\beta = \begin{cases} 0 & \text{if } \mu(\phi) > 0 \\ b * m_{\langle s \rangle} * \widehat{\overline{m}} & \text{if } \mu(\phi) \leq 0 \end{cases}$$

where $m_{\langle s \rangle}$ is the counting measure on the subgroup $\langle s \rangle \cong \mathbb{Z}$ and $\overline{m}$ is the unique Radon $\widehat{\mu}$-invariant measure on $\mathrm{Hor}(\Gamma)$ which invariant by right action of K_α (we denote by $\widehat{\nu}$ the image of the measure ν by the inversion on the group and the convolution product is calculated considering all measures as measures on Γ).

We observe the measure $\overline{m}$ is finite if and only if $\mu(\phi) < 0$; in this case the total mass is $-\frac{1}{\mu(\phi)}$.

The hypothesis that μ is spread out is not necessary when Γ is a subgroup of $\mathrm{Aff}(\mathbb{T})$ that acts in sufficiently regular way and, in particular, when the random walk is supported by $\mathrm{Aff}(\mathbb{Q}_p)$ or by the Lamplighter group and we look at the limits towards a point of $\partial^*\mathbb{T}$.

The characterization of the limits of the potential kernel is the starting point for a more detailed study of random walks on $\mathrm{Aff}(\mathbb{T})$ that is carried out in [2]. Woess [18] has identified the Martin boundary of a random walk on a transitive subgroup of the tree isometries with the tree's boundary. Using this results, it was proved that, if the measure μ has a compact support and continuous density, it is possible to give an integral representation of μ-invariant measures by mean of the measures

ν_α. For instance if $\mu(\phi) = 0$, then every μ-invariant measure ν on Γ can be written in a unique way as

$$\nu = c_\nu m_\Gamma^r + \int_{\partial^*\mathbb{T}} \nu_\beta \, \eta_\nu(d\beta)$$

where m_Γ^r is the right Haar measure of Γ. By this integral representation, in turn, one can prove uniqueness of the measure on Γ that is μ-invariant on the right and the left, and, via ratio limit theorems of Guivarc'h [7], a local limit theorem, as in the real case (cf.[11]). Under suitable moment and regularity conditions, when $\mu(\phi) = 0$, one can deduce that

$$\lim_{n\to\infty} c(n) n^{3/2} \mu^{(n)} = \overline{m}' * m_{\langle s\rangle} * \widehat{\overline{m}}$$

where $\overline{m}'$ is the unique μ-invariant measure on $\mathrm{Hor}(\Gamma)$ which is invariant by right action of K_α, and $c(n)$ is a sequence bounded away from 0 and $+\infty$, that can be proved to be constant when $\Gamma = \mathrm{Aff}(\mathbb{Q}_p)$.

We would like to conclude this section giving the guiding line of our study.

One of our main tools is a renewal equation, which says that there exists a probability measure ρ such that

$$\rho * U = \nu_\alpha \quad \text{on } \{g \in \Gamma \mid \phi(g) \leq 0\}.$$

Namely, when the starting point of the right random is distributed according to ρ, its potential measure is given by the limit measure, at least on "half" of the group. This equality is proved, after some preliminary results, at the end of the next section.

The second main step is to determine several fundamental invariance properties for the accumulation points of the potential kernel. In particular we need to show that these accumulation points are left invariant:

$$\lim_{n\to\infty} g_n * U = \lim_{n\to\infty} g_n g * U \qquad \forall g \in \Gamma$$

whenever the limit exists. This is a necessary condition to show that the potential kernel can be continuously extended to the tree's boundary, because it is easily checked that whenever $g_n \to \alpha$ then also $g_n g \to \alpha$. It is at this place where we encounter the main differences from the real case and where the hypothesis that μ is spread out become necessary. In section 3 we show this and other regularity properties and we give the proofs of our main results.

2. Preliminary results and a renewal equality

In this section we are going to give some preliminary results. In the first subsection we describe the convergence of the right random walk to the boundary improving a result originally due to Cartwright, Kaimanovich and Woess [4]. Next we analyze the action of the random affinities on the bottom boundary of the tree, $\partial^*\mathbb{T}$. Finally we provide a renewal equality, for both the action on the boundary and the random walk on the group, that will serve as one of our main tools. For this last result we use similar methods as those that have been used for the study of random walks of the real affine group by Babillot, Bougerol and Elie in [1].

2.1. Convergence of the random walk to the boundary.

Since the closed subgroup generated by the support of the law μ is non-exceptional, the random walks L_n and R_n are transient, so that the accumulation points of their trajectories lie on the boundary of the tree. Namely, the right random walk converges to a random variable on the boundary $\partial\mathbb{T}$.

Theorem 2.1. *Suppose that* $\mathbb{E}[|\phi(X_1)|] < \infty$.

(1) *If* $\mu(\phi) < 0$ *then* $R_n \to \omega$ *almost surely.*

(2) *If* $\mu(\phi) > 0$ *and* $\mathbb{E}[|X_1|] < \infty$ *then* $R_n \to \xi_\infty$ *almost surely, where* ξ_∞ *is a random element in* $\partial^*\mathbb{T}$. *The law* m *of* ξ_∞ *is supported by* $\partial^*\Gamma$ *and carries no point mass.*
(3) *If* $\mu(\phi) = 0$ *and* $\mathbb{E}[|X_1|] < \infty$ *then* $R_n \to \omega$ *almost surely.*

Proof. Results (1) and (2) are in [4], who proved (3) only under an exponential moment condition. Our proof of (3) uses a method developed in [3].

For all $x \in \mathbb{T}$, consider the cone

$$C_x = \{y \in \mathbb{T} : x \succeq y\}.$$

We show that for every $x \in \mathbb{T}$

$$\mathbb{P}[R_n o \in C_x \text{ infinitely often }] = 0.$$

Let m_Γ^r be the right Haar measure of Γ. First, we prove that for all $x \in \mathbb{T}$ and m_Γ^r-almost all $g \in \Gamma$

$$\mathbb{P}[gR_{n+1}o \in C_x,\, gR_n o \notin C_x \;\text{ infinitely often }] = 0.$$

Using the Borel-Cantelli Lemma, it is sufficient to show that

$$\sum_{n=0}^{\infty} \mathbb{P}[gR_{n+1}o \in C_x,\, gR_n o \notin C_x] = g * U(\psi) < +\infty$$

where $\psi(g) = \mathbb{P}[gX_1 o \in C_x,\, go \notin C_x]$, and using Lemma 2.2 in [3] we just need to show that ψ is m_Γ^r-integrable.

Observe that

$$\begin{aligned}\int_\Gamma \psi(g) m_\Gamma^r(dg) &= \mathbb{E}\Big[\int_\Gamma 1_{[gX_1 o\in C_x,\, go\notin C_x]} m_\Gamma^r(dg)\Big]\\ &= \mathbb{E}\Big[\int_\Gamma 1_{[go\in C_x, gX_1^{-1}o\notin C_x]} m_\Gamma^r(dg)\Big]\end{aligned}$$

and that

$$\begin{aligned}\{g\in\Gamma :\, go\in C_x, gX_1^{-1}o\notin C_x\} &= \{g\in\Gamma :\, go\in C_x, g(o\wedge X_1^{-1}o)\notin C_x\}\\ &= \bigcup_{y\in S} V(y\to o)\end{aligned}$$

where $V(y \to o) = \{g \in \Gamma : gy = o\}$ and S is the geodesic segment that joins o to the vertex immediately before $o \wedge X_1^{-1}o$ (or the empty set if $o \wedge X_1^{-1}o = o$). Observe that for any affinity γ such that $\gamma y = o$ one has $V(y \to o) = V(o \to o)\gamma$, thus $m_\Gamma^r(V(o \to o)) = m_\Gamma^r(V(y \to o))$ for all y. Since the segment S contains exactly $-\phi(o \wedge X_1^{-1}o)$ vertices, we have

$$\int_\Gamma 1_{[go\in C_x, gX_1^{-1}o\notin C_x]} m_\Gamma^r(dg) \le -\phi(o\wedge X_1^{-1}o) m_\Gamma^r(V(o\to o))$$

so that

$$\int_\Gamma \psi(g) m_\Gamma^r(dg) \le \mathbb{E}\big[-\phi(o\wedge X_1^{-1}o)\big]\, m_\Gamma^r(V(o\to o)) \le \mathbb{E}[|X_1|]\, m_\Gamma^r(V(o\to o)) < +\infty.$$

We proved that for almost all g, almost surely, $gR_n o$ cannot pass from C_x^c to C_x but a finite number of times.

As the set $V(x \to x)$ is open and thus has strictly positive Haar measure, there exists $g \in V(x \to x)$ such that the event

$$\begin{aligned}[gR_{n+1}o \in C_x,\, gR_n o \notin C_x] &= \big[R_{n+1}o \in g^{-1}C_x,\, R_n o \notin g^{-1}C_x\big]\\ &= [R_{n+1}o \in C_x,\, R_n o \notin C_x].\end{aligned}$$

takes place only a finite number of times.

On the other hand the vertex set of the tree is countable, whence almost surely

$$\forall x \in \mathbb{T} :\; R_{n+1}o \in C_x,\, R_n o \notin C_x \quad \text{a finite number of times.}$$

If $\mu(\phi) = 0$, the random walk $\phi(R_n)$ is recurrent on $\mathbb{Z}$, and visits the interval $]-\infty, \phi(x)]$ infinitely often. In particular $R_n o \in C_x^c$ infinitely often. As R_n cannot go back to C_x but a finite number of times, we conclude that it is in C_x^c for all sufficiently large n . □

2.2. **Action on $\partial^*\mathbb{T}$.** As we have noted, the affine group has a natural action on the boundary of the tree that may also be considered as a boundary for the group itself; the behavior of the Markov chain on $\partial^*\mathbb{T}$ provides then useful information for the study of the random walk on the group itself.

The analogue of this Markov chain in the setting of real affine group is the process induced on $\mathbb{R}$ by the natural action. We may note that while in the real case the joint behavior of this chain and of the homothetic component describes completely the random walk, this does not hold on the tree because the process on the group is more complex. In fact, $\mathrm{Aff}(\mathbb{T})$ is the semi-direct product of $\mathbb{Z}$ and of the horocyclic group $\mathrm{Hor}(\mathbb{T})$, which is "bigger" than the boundary of the tree.

To understand the behavior of the random walk on the group we also need to analyze the process induced on the horocyclic group by the action of $\mathrm{Aff}(\mathbb{T})$. Nevertheless the techniques we are going to use cannot be applied directly in this setting, mainly because the action of $\mathbb{Z}$ on the horocyclic group is not sufficiently contractive. However at the end of this section we will be able to deduce, as a corollary, some results in this context as well.

Let Υ_0 be a random variable defined on the probability space $(\Omega, \mathbb{P})$, with value in $\partial^*\mathbb{T}$, independent from the increments $\{X_n\}_n$ of the random walk. The Markov chain induced on $\partial^*\mathbb{T}$ is the process

$$\Upsilon_k = L_k \Upsilon_0 = X_k \cdots X_1 \Upsilon_0.$$

Its transition kernel is

$$Pf(v) = \mathbb{E}[f(X_1 \cdot v)] = \int_{\mathrm{Aff}(\mathbb{T})} f(gv)\mu(dg) = \mu \circledast v(f). \tag{2.1}$$

Here and in the sequel, the symbol $\circledast$ denotes the convolution of a measure on the group Γ and a measure on a Γ-space.

The proofs of some results of this section are formally very similar to the analogues in the real case. We have translated them to the present setting for reader's convenience.

The behavior of this chain is directly related to the random walk $S_n = \phi(L_n)$ on $\mathbb{Z}$, that contains the information on how the random affinities contract or dilate the boundary.

When $\mu(\phi) = \mathbb{E}[\phi(X_1)] \neq 0$, one may directly obtain properties of transience or recurrence of the induced Markov chain from what is known for the process on the group.

Proposition 2.2. *Suppose that* $\mathbb{E}[|X_1|] < \infty$.

(1) *If* $\mu(\phi) < 0$ *then for all* $v \in \partial^*\mathbb{T}$

$$\lim_{n\to\infty} L_n v = \omega \qquad \textit{almost surely}$$

and the chain $\{\Upsilon_n\}_n$ *is transient.*

(2) *If* $\mu(\phi) > 0$, *the law* m *of the random variable* $\xi_\infty = \lim_{n\to\infty} R_n$ *is the unique stationary probability measure for the Markov chain on* $\partial^*\mathbb{T}$, *thus* $\{\Upsilon_n\}_n$ *is positive recurrent. Whence, almost surely, for all* v *in* $\partial^*\mathbb{T}$, *the sequence* $\{L_n v\}_n$ *visits infinitely often every open set of* $\partial^*\mathbb{T}$ *of non zero* m*-measure.*

Proof. (1) The distance between $L_n v$ and a fixed end α of $\partial^*\mathbb{T}$ is given by

$$\Theta(L_n v, \alpha) = q^{-\phi(L_n)}\Theta(v, L_n^{-1}\alpha).$$

As $\mu(\phi) < 0$, the random walk $\phi(L_n)$ on $\mathbb{Z}$ converges almost surely to $-\infty$. On the other hand Theorem 2.1 states that the right random walk

$$L_n^{-1} = X_1^{-1} \cdots X_n^{-1} = \hat{R}_n,$$

whose drift is $\mathbb{E}\big[\phi(X_1^{-1})\big] = -\mu(\phi) < 0$, converges to a random element $\hat{\xi}_\infty$ of $\partial^*\mathbb{T}$, whose law does not charge any point. Thus, for every end v in $\partial^*\mathbb{T}$, there exists a sub-set $\Omega_v \subseteq \Omega$ of measure 1 such that on Ω_v:

$$\lim_{n\to\infty} \phi(L_n) = -\infty \quad \text{and} \quad \lim_{n\to\infty} \hat{R}_n = \hat{\xi}_\infty \neq v;$$

therefore on Ω_v

$$\lim_{n\to\infty} \Theta(L_n v, \alpha) = +\infty.$$

(2) The probability measure m is μ-invariant. In fact, let f be a bounded continuous function on $\partial^*\mathbb{T}$ and let X be a random variable on $\mathrm{Aff}(\mathbb{T})$ of law μ independent from the sequence $\{X_n\}_{n\geq 1}$ then

$$\begin{aligned} \mu * m(f) &= \mathbb{E}[f(X\xi_\infty)] = \mathbb{E}\Big[f(X \lim_{n\to\infty} X_1 \cdots X_n)\Big] \\ &= \mathbb{E}\Big[f(\lim_{n\to\infty} X X_1 \cdots X_n)\Big] \quad \text{as } X \text{ acts conituously on } \mathrm{Aff}(\mathbb{T}) \cup \partial^*\mathbb{T} \\ &= \mathbb{E}[f(\xi_\infty)] = m(f) \end{aligned}$$

If m' is another invariant probability measure and Υ_0 a random variable on $\partial^*\mathbb{T}$ with law m', independent from the increments $\{X_n\}_n$, then for every bounded continuous function f on $\partial^*\mathbb{T}$

$$\begin{aligned} \eta(f) &= \mathbb{E}[f(L_n\Upsilon_0)] = \mathbb{E}[f(R_n\Upsilon_0)] \\ &= \lim_{n\to\infty} \mathbb{E}[f(R_n\Upsilon_0)] = \mathbb{E}\Big[\lim_{n\to\infty} f(R_n\Upsilon_0)\Big] = \mathbb{E}[f(\xi_\infty)] = m(f) \end{aligned}$$

by dominated convergence, Theorem 2.1 and since

$$\lim_{g\to\xi} gv = \xi \qquad \text{for all } v, \xi \in \partial^*\mathbb{T}.$$

Thus there is a unique invariant probability measure, and, by the ergodic theorem, for m-almost every v, the Markov chain $L_n v$ visits infinitely often every set of positive m-measure. Furthermore, for open sets one can assure that this holds for a every starting point v (and not only for almost all) because of the contracting property of the Markov chain:

$$\lim_{n\to\infty} \Theta(L_n v, L_n \varsigma) = \lim_{n\to\infty} q^{-\phi(L_n)}\Theta(v, \varsigma) = 0. \tag{2.2}$$

□

A classical technique to deal with the centered case, $\mu(\phi) = 0$, is to extract from the original chain a sub-chain with positive drift, to which one can apply the previous results. We consider the sequence of ladder stopping times l_k giving the times when the random walk $S_n = \phi(L_n)$ on $\mathbb{Z}$ reaches a new maximum

$$l_k = \min\{n > l_{k-1} : \ S_n > S_{l_{k-1}}\} \quad \text{and} \quad l_0 = 0.$$

The process obtained regarding the left random walk at these times

$$L_{l_k} = \left(X_{l_k} \cdots X_{l_{k-1}+1}\right) L_{l_{k-1}}$$

is still a left random walk, whose law, that is the law of L_{l_1}, will be denoted by μ_l. For convenience we also set $l_1 = l$.

The drift of this random walk is clearly positive (maybe infinite), $\mathbb{E}[\phi(L_l)] > 0$, but to apply the previous results one has to make sure that its law is integrable and for this we need a moment of order $2 + \varepsilon$.

Lemma 2.3. *The closed group generated by μ_l coincides with the group generated by the support of μ, and if*

$$\mathbb{E}\Big[\phi(X_1)^2 + |b(X_1)|^{2+\varepsilon}\Big] < +\infty$$

then

$$\mathbb{E}[|L_l|] < +\infty.$$

Proof. This is proved in [4] (Proposition 4), in the equivalent version regarding right random walks and decreasing ladder times. □

Under this stronger moment hypothesis, we can apply to the random walk L_{l_n} the results of the previous proposition, in particular to ensure that the Markov chain Υ_{l_n} has a unique invariant probability measure that is denoted m_l. We get then the following results:

Proposition 2.4. *Suppose that μ has a moment of order $2+\varepsilon$ and that $\mu(\phi) = 0$. Then the chain Υ_n is recurrent, in the sense that almost surely for all υ in $\partial^*\mathbb{T}$ the chain $L_n\upsilon$ visits infinitely often every open set of non-zero m_l-measure. Furthermore, there exists a unique μ-invariant Radon measure on $\partial^*\mathbb{T}$.*

Proof. Applying to $L_{l_n}\upsilon$ the results of Proposition 2.2, we obtain that $L_{l_n}\upsilon$ visits infinitely often every open set of non-zero m_l-measure, whence, a fortiori, the same holds for $L_n\upsilon$.

Thus for every $\upsilon \in \partial^*\mathbb{T}$ and any non-negative continuous function f on $\partial^*\mathbb{T}$ such that $m_l(f) \neq 0$

$$\sum_{n=0}^{+\infty} P^n f(\upsilon) = \mathbb{E}[f(L_n \cdot \upsilon)] = +\infty$$

and the chain is topologically conservative. As P is a Feller operator, by [12] (Theorem 5.1), the chain Υ_n has an invariant Radon measure.

In the next lemma we shall prove what is known as local contraction property. Using this result and Chacon-Ornstein Theorem one obtains uniqueness of the invariant measure along the same lines as in [3], Theorem 3. □

In the centered case the random walk L_n has neither a contracting or dilating action, as the distance between two trajectories

$$\Theta(L_n\upsilon, L_n\varsigma) = q^{-\phi(L_n)}\Theta(\upsilon, \varsigma)$$

does not converge to zero as in the case of positive drift , nor to $+\infty$ as in the case of negative drift, but oscillates between these two extremes. However, if we do not look globally at this process but only through a compact window, we can recover a stability property:

Lemma 2.5. *If the measure μ has a first moment and if $\mu(\phi) = 0$, for every compact set K in $\partial^*\mathbb{T}$ and for every pair of ends $\upsilon, \varsigma \in \partial^*\mathbb{T}$, almost surely*

$$\lim_{n\to\infty} \Theta(L_n\upsilon, L_n\varsigma)\, 1_K(L_n\upsilon) = \lim_{n\to\infty} \Theta(\upsilon, \varsigma)\, q^{-\phi(L_n)} 1_K(L_n\upsilon) = 0. \tag{2.3}$$

Proof. The local contraction (2.3) is a direct consequence of the fact that, in the centered case, the right random walk on $\mathrm{Aff}(\mathbb{T})$ converges to the mythical ancestor ω.

Indeed, suppose that (2.3) does not hold. Then there exists a cone with vertex $y \in \mathbb{T}$

$$C_y = \{x \in \mathbb{T} \cup \partial^*\mathbb{T} : x \succeq y\}$$

and an integer M such that, with probability 1,

$$L_n\upsilon \in C_y \text{ and } \phi(L_n) < M \quad \text{for infinitely many } n.$$

Then for any x in the geodesic $\overline{v\omega}$ such that $\phi(x) < \phi(y) - M$ one has

$$\begin{aligned} 1 = \mathbb{P}[L_n x \in \overline{y\omega} \text{ infinitely often}] &= \mathbb{P}\left[x \in \overline{(L_n^{-1}y)\,\omega} \text{ infinitely often}\right] \\ &\leq \mathbb{P}\left[L_n^{-1}y \in C_x \text{ infinitely often}\right]. \end{aligned}$$

On the other hand

$$L_n^{-1} = X_n^{-1} \cdots X_1^{-1} = \hat{R}_n$$

is a right random walk with first moment and zero drift. Thus we have obtained a contradiction, because we know by Theorem 2.1 that $L_n^{-1} = \hat{R}_n \to \omega$ almost surely. □

As in the real case, it is possible to construct the unique invariant Radon measure, m, using the invariant probability measure, m_l, of the contracting sub-chain, in the following way:

$$m(f) = \frac{1}{\mathbb{E}[S_l]} \int_{\partial^*\mathbb{T}} \mathbb{E}\left[\sum_{k=0}^{l-1} f(L_k \cdot v)\right] m_l(dv) \tag{2.4}$$

Using the strong Markov property one can see that m is μ-invariant. Observe that the stopping time l is not integrable (cf. [16] Chapter IV.18), so that the measure m does not have finite mass. The fact that it is finite on compact sets is not evident and it will proved in the next sub-section in Corollary 2.7.

The measure (2.4) can also be defined if the step law of the random walk has first moment and the drift is positive. In this case the time l is integrable and m has total mass equal to

$$m(\partial^*\mathbb{T}) = \frac{\mathbb{E}[l]}{\mathbb{E}[S_l]} = \frac{\mathbb{E}[l]}{\mathbb{E}[l]\,\mathbb{E}[S_1]} = \frac{1}{\mu(\phi)}.$$

by Wald's equality (cf. [5], page 350). This measure is then just a normalization of the unique invariant probability measure.

2.3. **A renewal equality.** In order to obtain the announced renewal equality we consider the joint action of random affinities on both the bottom boundary and the integers. In other words the result concerns the Markov chain on $\partial^*\mathbb{T} \times \mathbb{Z}$ whose transition kernel is

$$\tilde{P}f(v,z) = \mu \circledast (v,z)(f), \quad \text{where } \mu \circledast (v,z)(f) = \mathbb{E}[f(X_1 v, \phi(X_1) + z)] \tag{2.5}$$

and whose potential kernel is

$$\sum_{n=0}^{\infty} \tilde{P}^n f(v,z) = U \circledast (v,z)(f)$$

where $U = \sum_{n=0}^{\infty} \mu^{(n)}$ is the potential measure of the random walk on the group.

Proposition 2.6. *Assume that μ has a moment of first order and $\mu(\phi) > 0$ or that μ has a moment of order $2+\varepsilon$ and $\mu(\phi) = 0$. Let m be the measure defined in (2.4). Then there exists a probability measure p on $\partial^*\mathbb{T} \times \mathbb{Z}$ such that for every non-negative function f with support in $\partial^*\mathbb{T} \times \mathbb{Z}_+$*

$$U \circledast p(f) = m \times m_{\mathbb{Z}}(f) \tag{2.6}$$

where $m_{\mathbb{Z}}$ is the counting measure on $\mathbb{Z}$.

Proof. For the reader's convenience, we sketch the proof that formally follows the same scheme as Proposition 2.1 in [1].

We first show that there exists a probability p on $\partial^*\mathbb{T} \times \mathbb{Z}$ such that

$$U_l \circledast p = \frac{1}{\mathbb{E}[S_l]} \left(m_l \times 1_{[0,+\infty[} m_{\mathbb{Z}}\right) \tag{2.7}$$

where $U_l = \sum_{n=0}^{\infty} \mu_l^{(n)}$. Let $\widetilde{\nu}_l = \left(m_l \times 1_{[0,+\infty[} m_{\mathbb{Z}}\right)$. We observe that for every measurable non-negative function f :

$$\begin{aligned}
\mu_l \circledast \widetilde{\nu}_l(f) &= \int_{\partial^*\mathbb{T}\times\mathbb{Z}} \mathbb{E}\left[f(L_l v, S_l + z) 1_{[z\geq 0]}\right] m_l(dv) m_{\mathbb{Z}}(dz) \\
&= \int_{\partial^*\mathbb{T}\times\mathbb{Z}} \mathbb{E}\left[f(L_l v, z) 1_{[z-S_l\geq 0]}\right] m_l(dv) m_{\mathbb{Z}}(dz) \\
&\leq \int_{\partial^*\mathbb{T}\times\mathbb{Z}} \mathbb{E}\left[f(L_l v, z) 1_{[z\geq 0]}\right] m_l(dv) m_{\mathbb{Z}}(dz) \\
&= \widetilde{\nu}_l(f) \quad \text{because } m_l \text{ is } \mu_l - \text{invariant.}
\end{aligned}$$

Thus

$$p' := \widetilde{\nu}_l - \mu_l \circledast \widetilde{\nu}_l$$

is a positive measure and one calculates its total mass $\mathbb{E}[S_l]$. Furthermore, for every bounded non-negative function $f = f_1 \times f_2$ such that f_2 has compact support

$$\lim_{n\to\infty} \mu_l^{(n)} \circledast \widetilde{\nu}_l(f) \leq \lim_{n\to\infty} \|f_1\|_\infty \mathbb{E}\left[\int_{S_{l_n}}^{+\infty} f_2(z) m_{\mathbb{Z}}(dz)\right] = 0.$$

Thus

$$U_l \circledast p'(f) = \lim_{n\to\infty} \sum_{k=0}^{n} \left(\mu^{(k)} \circledast \widetilde{\nu}_l - \mu^{(k)} \circledast \widetilde{\nu}_l\right)(f) = \widetilde{\nu}_l$$

and therefore the probability measure $p = \dfrac{p'}{\mathbb{E}[S_l]}$ verifies (2.7) on compact sets and thus everywhere.

Let f be a non-negative function with support in $\partial^*\mathbb{T} \times \mathbb{Z}_+$. To conclude, one has to apply (2.7) to the non-negative Borel function $F(\xi, z) = \mathbb{E}\left[\sum_0^{l-1} f(L_k v, S_k + z)\right]$ and check that

$$U \circledast p(f) = U_l \circledast p(F) = \widetilde{\nu}_l(F) = m \times m_{\mathbb{Z}}(f).$$

□

As announced, a direct consequence of this result is the following

Corollary 2.7. *Under the hypothesis of the previous proposition, the measure m defined in (2.4) is a Radon measure and coincides with the unique μ-invariant Radon measure on $\partial^*\mathbb{T}$.*

Proof. As the random walk on the affine group is transient, its left potential kernel $U * g(f)$ is bounded for every bounded function f with compact support. As the group $\Gamma = \mathrm{Hor}(\Gamma) \rtimes \mathbb{Z}$ is a compact extension of $\partial^*\mathbb{T} \times \mathbb{Z}$, it is easy to see that also the potential kernel of the chain induced on $\partial^*\mathbb{T} \times \mathbb{Z}$, that is

$$U \circledast x(f) = \int_\Gamma f(g \cdot x) U(dg)$$

is bounded in $x \in \partial^*\mathbb{T} \times \mathbb{Z}$ for any bounded compactly supported function f.

Let K be a compact subset of $\partial^*\mathbb{T}$. Then, as p is a probability, one has

$$m(K) = m \times m_{\mathbb{Z}}(1_K \times 1_{\{0\}}) = \int_{\partial^*\mathbb{T}\times\mathbb{Z}} U \circledast x(1_K \times 1_{\{0\}}) p(dx) < +\infty$$

□

2.4. **The action on** $\mathrm{Hor}(\Gamma)$ **and a renewal equality on the group.** To understand the random walk on the group Γ, we are also interested on the process induced on the horocyclic group by the action defining the semi-direct product $\Gamma = \mathrm{Hor}(\Gamma) \rtimes_s \mathbb{Z}$, that is

$$g \cdot b := b(g)s^{\phi(g)}bs^{-\phi(g)} = gbs^{-\phi(g)}. \tag{2.8}$$

Even if the techniques that we have used are not completely adapted to this setting, we can easily deduce some useful results.

As we have observed, the bottom boundary of the tree is homeomorphic to the quotient of the horocyclic group by the compact stabilizer of an end α, that is $\partial^*\Gamma = \mathrm{Hor}(\Gamma)/K_\alpha$. It is then possible to extend any measure, m, on $\partial^*\Gamma$ to a measure, $\overline{m}$, on $\mathrm{Hor}(\Gamma)$ by setting

$$\overline{m}(f) = \int_{\partial^*\Gamma\times K_\alpha} f(xk)m_{K_\alpha}(dk)m(dx) \tag{2.9}$$

where m_{K_α} is the Haar measure of K_α. By construction $\overline{m}$ is right invariant by the right action of the rotations K_α and, above all, it is well adapted to the convolutions induced by actions of the group Γ on the boundary and on the horocyclic group, defined in (2.1) and (2.8) respectively. In fact for every affinity g, one has

$$g \circledast \overline{m} = \overline{g \circledast m}.$$

Thus, whenever one has an invariant measure on the boundary, it is possible to extend it to an invariant measure on the horocyclic group (right invariant by action of K_α). It is also possible to extend the renewal equality of Proposition 2.6 to $\mathrm{Hor}(\Gamma) \times \mathbb{Z}$, in order to get a renewal equality on the whole group. Summarizing, we obtain the following

Corollary 2.8. *Assume that μ has a moment of first order and $\mu(\phi) > 0$ or that μ has a moment of order $2+\varepsilon$ and $\mu(\phi) = 0$. Then there exists a μ-invariant Radon measure, $\overline{m}$, on $\mathrm{Hor}(\Gamma)$ obtained by extension of the invariant measure on $\partial^*\Gamma$. This measure has finite mass if and only if $\mu(\phi) > 0$.*

If $\overline{m}$ is normalized in such a way that it is the extension of the measure defined (2.4), then there exists a probability $\overline{p}$ on Γ such that

$$U * \overline{p} = \overline{m} * m_{\langle s\rangle} \quad \text{on } \{g \in \Gamma \,|\, \phi(g) \geq 0\}, \tag{2.10}$$

where $m_{\langle s\rangle}$ is the counting measure on the group $\langle s\rangle \cong \mathbb{Z}$ and $*$ is the convolution for measures on the group Γ.

Proof. Just observe that equation (2.10) can be obtained proceeding as in Proposition 2.6, where we never really used the structure of the chain on the boundary, but just the fact that there exists a probability measure for the contracting sub-chain. Otherwise, we can obtain (2.10) by extension from its equivalent in the boundary case by setting

$$\overline{p}(f) = \int_{\partial^*\mathbb{T}\times K_\alpha\times\mathbb{Z}} f(xks^h)m_{K_\alpha}(dk)p(dx\,dh).$$

□

The question that remains still open is whether the μ-invariant measure defined by extension is the unique invariant measure, or, in other words, if all invariant measures are invariant by K_α. In the case of p-adic affine group, it seems likely to have a positive answer, that could be obtained regarding the random walk on the group of rotations obtained by projection. In the general case there is no such canonical projection on K_α and the solution do not seem to be so evident.

Remark. As we did not impose continuity conditions on the law of the random walk, we can apply these results to products of random affine transformations with rational coefficients, considered as random walks on the group $\mathrm{Aff}(\mathbb{Q}_p)$. One obtains the following

Corollary 2.9. *Let μ be a measure on $\mathrm{Aff}(\mathbb{Q}) = \mathbb{Q} \rtimes \mathbb{Q}^*$. Let t and a be the projections of $\mathrm{Aff}(\mathbb{Q})$ on $\mathbb{Q}$ and $\mathbb{Q}^*$ respectively. Assume the measure μ is irreducible on $\mathrm{Aff}(\mathbb{Q}_p)$ i.e.*

$$\mathbb{P}\Big[|a(X_1)|_p = 1\Big] < 1 \quad \text{and} \quad \forall y \in \mathbb{Q}: \quad \mathbb{P}[a(X_1)y + t(X_1) = y] < 1$$

and that the moment conditions of Propositions 2.2 and 2.4 are satisfied.

If $\mathbb{E}\Big[\log |a(X_1)|_p\Big] \leq 0$, then there exists a unique μ-invariant Radon measure on $\mathbb{Q}_p$, that has finite mass if and only if $\mathbb{E}\Big[\log |a(X_1)|_p\Big] < 0$.

It is easily checked the measure $\overline{m} \times m_{\mathbb{Z}}$ is left μ-invariant on the group. Thus for every integrable function f, the functions

$$h((t,a)) = (t,a) * (\overline{m} \times m_{\mathbb{Z}})(f)$$

are non-trivial right harmonic on $\mathrm{Aff}(\mathbb{Q})$, and they are bounded if $\mathbb{E}\Big[\log |a(X_1)|_p\Big] < 0$ and f is bounded.

The group of affine transformations with rational coefficients is usually regarded as a dense subgroup of the group of affine transformations with real coefficients, $\mathrm{Aff}(\mathbb{R})$. However, generic random walks on $\mathrm{Aff}(\mathbb{Q})$ do not always behave like random walks that are completely adapted to the topology of $\mathrm{Aff}(\mathbb{R})$, like those whose law is spread out on the latter group. For instance it is known that in this last case when $\mathbb{E}[\log |a(X_1)|] < 0$, there is no bounded harmonic function, i.e the Poisson boundary is trivial. On the other hand it has been shown ([10]) that for random walks on the affine group of dyadic integers such that

$$\mathbb{E}[\log |a(X_1)|] = -\mathbb{E}[\log |a(X_1)|_2] < 0$$

the Poisson boundary is not-trivial and coincides with $\mathbb{Q}_2$. The last corollary shows that every random walk on $\mathrm{Aff}(\mathbb{Q})$ whose p-adic drift is negative (for some p) has a non trivial Poisson boundary (independently from the real drift). It seems likely that a complete description of the Poisson boundary of the random walk on $\mathrm{Aff}(\mathbb{Q})$ can be obtained by embedding this group simultaneously in the real affine group and in all the p-adic ones.

3. Limit measures of the potential kernel

This section is devoted to the study of the limit measures of the potential kernel near the boundary of the group. W.Woess [18] studied the asymptotic behavior of the Martin kernel (i.e. the normalization of the potential kernel) for a random walk, with continuous and compactly supported density, on any closed and transitive group of isometries of homogeneous tree and showed that the Martin boundary can be identified with $\partial\mathbb{T}$. For the potential kernel of random walks on $\mathrm{Aff}(\mathbb{T})$ it is possible to obtain the same kind of results of continuous extension to the geometrical boundary without asking for a compact support and moreover to obtain the form of the limit measures in terms of the invariant measure on the boundary and of the counting measure on $\mathbb{Z}$.

In comparison with Elie's work [6] on Lie groups, by which our study was originally inspired, we have to face, as we announced, some new phenomena of non-commutativity and the lack of stiffness of the tree structure. The use of the renewal equality introduced in the last section enables us to appreciably simplify the proofs and, mostly, to avoid continuity hypotheses for random walks on sufficiently regular groups, as $\mathrm{Aff}(\mathbb{Q}_p)$, when one looks to the limit of the potential kernel towards a point of $\partial^*\mathbb{T}$.

It is also worth observing that all the conclusions of this paper can be obtained in the same way for the real case. In particular we can improve Elie's results assuring that even if the measure is not spread out, the associated potential kernel on on the real affine group, $(t,a)*U$, converges to a limit measure whenever (t,a) converges to $(t_0,0)$.

In our study, we will need the following general results on uniform continuity of the potential kernel (cf. [6], Proposition 2.7 and Theorem 2.9)

Lemma 3.1. *Let $\{g_n\}_n$ be a sequence of elements in Γ such that $\{g_n*U\}_n$ converges vaguely.*

1. Left continuity. *There exists a subsequence $\{g_{n_k}\}_k$ such that, for every $y\in\Gamma$, the sequence of measures $\{yg_{n_k}*U\}_k$ converges vaguely. Furthermore, for all $f\in C_c(\Gamma)$, the sequences $\{yg_{n_k}*U(f)\}_k$ converge uniformly when y varies in a compact set.*
2. Right continuity. *If μ is spread out, there exists a subsequence $\{g_{n_k}\}_k$ such that, for every $y\in\Gamma$, the sequence of measures $\{g_{n_k}y*U\}_k$ converges vaguely. Furthermore, for all $f\in C_c(\Gamma)$, the sequences $\{g_{n_k}y*U(f)\}_k$ converge uniformly when y varies in a compact set.*

We observe that left continuity is almost straightforward, while right continuity is a more subtle phenomenon, that requires a stronger regularity condition (the measure is spread out). We will often use this Lemma to guarantee that when we perturb the sequence g_n (to the right or to the left) by a sequence y_n that converges to y, then on a subsequence the limit does not change if we replace y_n by y; for instance

$$\lim_{k\to\infty} g_{n_k}y_{n_k}*U(f)=\lim_{k\to\infty} g_{n_k}y*U(f). \tag{3.1}$$

As the structure of the limit measures in a neighborhood of ω differs from the structure of the limit measure in the neighborhood of a point in the bottom boundary $\partial^*\Gamma$, we will study the two cases in two different sub-sections.

3.1. **The limits of the potential kernel at $\partial^*\mathbb{T}$.** We start by showing some invariance properties for the accumulation point of the potential kernel. In a second step, using these results and the renewal formula of the previous section, we show that it is possible to extend the potential kernel by continuity to $\partial^*\Gamma$.

3.1.1. *Invariance properties.* The structure of the limit measures depends not only on how the affinities act on the vertices at finite distance, but also on how they act near the boundary of the tree. Observe that while the first of these actions is completely adapted to the topology of pointwise convergence on the tree, the second action is not at all related to it: one can construct a sequence of elements of $\mathrm{Aff}(\mathbb{T})$ that converge to the identity, but such that they act non trivially on a sequence of sets sufficiently far away from the origin. For the subgroups of $\mathrm{Aff}(\mathbb{T})$ that have homogeneous action on the tree (as $\mathrm{Aff}(\mathbb{Q}_p)$) this problem does not occur, but for more general subgroups, to link the topology of the group and the action near the boundary will require the right continuity of the potential kernel (and therefore the hypothesis the μ is spread out).

The following lemmas are intended to clarify the behavior of an affinity near the boundary, and more precisely what happens if it is conjugates with a sequence of transformations that permits to explore how it acts faraway from an end α.

Lemma 3.2. *Let $\alpha\in\partial^*\mathbb{T}$ be a fixed end of the tree and let $s\in\Gamma$ be a reference homothety such that $s\alpha=\alpha$ and $\phi(s)=1$. Then, for every $g\in\Gamma$, the sequence*

$\{s^n g s^{-n}\}_{n\in\mathbb{N}}$ *is relatively compact in* Γ*, and all its accumulation points fix* α*. Furthermore, if* $g \in \mathrm{Hor}(\Gamma)$ *then the accumulation points belong to* K_α*.*

Proof. We first observe that $\{s^n g s^{-n}\alpha\}_n$ converges to the end α. In fact, since $s^n\alpha = \alpha$,

$$\lim_{n\to\infty} \Theta(\alpha, s^n g s^{-n}\alpha) = \lim_{n\to\infty} \Theta(s^n\alpha, s^n g\alpha) = \lim_{n\to\infty} q^{-n}\Theta(\alpha, g\alpha) = 0$$

Let α_m be the vertex of the geodesic $\overline{\alpha\omega}$ such that $\phi(\alpha_m) = m$. Since $\phi(s^n g s^{-n}) = \phi(g)$, for sufficiently large n and any integer m, the elements of the sequence $\{s^n g s^{-n}\}_n$ belong to the compact set

$$V(\alpha_m \to \alpha_{m+\phi(g)}) = \left\{\gamma \in \Gamma :\ \gamma\alpha_m = \alpha_{m+\phi(g)}\right\}.$$

Thus, every accumulation point of $\{s^n g s^{-n}\}_n$ belongs to

$$\bigcap_{m\in\mathbb{Z}} V(\alpha_m \to \alpha_{m+\phi(g)}) = \bigcap_{m\in\mathbb{Z}} s^{\phi(g)} V(\alpha_m \to \alpha_m) = s^{\phi(g)} K_\alpha$$

and it fixes the end α. □

If Γ is a subgroup that acts homogeneously on the tree, then the sequence $s^n g s^{-n}$ converges and its limit is given by the rotation and homothetic component of center α of g. For instance if $\Gamma = \mathrm{Aff}(\mathbb{Q}_p) = \mathbb{Q}_p^* \times \mathbb{Q}_p$, and we choose as reference homothety $s = (p, 0)$, then for $g = (a, x)$, the sequence $\{s^n g s^{-n}\}_n$ converges to $(a, 0)$. This does not hold for general subgroups, because the homothety s does not commute with the rotations, and this constitutes one of the main differences with the study of the renewal theory on Lie groups. L.Elie in [6] has in fact shown that any almost connected Lie groups whose potential kernel has an infinite number of accumulation points is the semi-direct product of a nilpotent Group (of "translations") and of the *direct* product of a compact Lie group (the rotations) and $\mathbb{R}$ (the homotheties).

The following lemma will enable us to get around the problem that for $b \in \mathrm{Hor}(\Gamma)$ the sequence $\{s^n b s^{-n}\}_n$ does not converge by giving an approximation of its accumulation points with sequences of the form $\{s^{n_k} r_k s^{-n_k}\}_k$, where $\{r_k\}_k$ is a sequence that converges to a rotation r. Roughly speaking r tells how b acts far away from α, but this information depends on the subsequence n_k (i.e. on the escape speed from α) and, in general, it is not related on how b behaves near α.

Lemma 3.3. *Let* $\mathbf{n} = \{n_k\}_{k\in\mathbb{N}}$ *be a sequence in* $\mathbb{Z}$ *such that* $\lim_{k\to\infty} n_k = +\infty$*. Then for every* $b \in \mathrm{Hor}(\Gamma)$ *there exists a sequence* $\{r_l\}_l$ *in* $\mathrm{Hor}(\Gamma)$ *that converges to a rotation* r *in* K_α *and a subsequence* $\mathbf{m} = \{m_l\}_{l\in\mathbb{Z}}$ *of* $\mathbf{n}$ *such that*

$$\lim_{l\to\infty} s^{m_l} b^{-1} r_l s^{-m_l} = e$$

Proof. By the preceding lemma one knows that there is a subsequence $\mathbf{n}'$ of $\mathbf{n}$ such that $\left\{s^{n'_k} b s^{-n'_k}\right\}_k$ converges and, therefore, is a Cauchy sequence, i.e.:

$$\lim_{k\to\infty} \left(s^{n'_k} b s^{-n'_k}\right)^{-1} s^{n'_{k+i}} b s^{-n'_{k+i}} = e$$

uniformly in $i \in \mathbb{N}$. Let $\{i(k)\}_{k\in\mathbb{N}}$ be a sequence of natural numbers such that $n'_{k+i(k)} - n'_k$ converges to $+\infty$, when k goes to $+\infty$. According to Lemma 3.2 the sequence

$$b_k = s^{n'_{k+i(k)} - n'_k} b s^{-(n'_{k+i(k)} - n'_k)}$$

is relatively compact, so that it is possible to extract a convergent subsequence $\left\{b_{k(l)}\right\}_{l\in\mathbb{N}}$. Let $r_l = b_{k(l)}$, let r be the rotation of center α obtained as limit

$$r = \lim_{l\to\infty} r_l = \lim_{l\to\infty} b_{k(l)}$$

and let $m_l = n'_{k(l)}$; then

$$\begin{aligned}\lim_{l\to\infty} s^{m_l} b^{-1} r_l s^{-m_l} &= \lim_{l\to\infty} s^{n'_{k(l)}} b^{-1}\left(s^{n'_{k(l)+i(k(l))}-n'_{k(l)}} b s^{-(n'_{k(l)+i(k(l))}-n'_{k(l)})}\right) s^{-n'_{k(l)}}\\ &= \lim_{l\to\infty}\left(s^{n'_{k(l)}} b s^{-n'_{k(l)}}\right)^{-1} s^{n'_{k(l)+i(k(l))}} b s^{-n'_{k(l)+i(k(l))}}\\ &= e.\end{aligned}$$

□

Using the last lemma along with the uniform continuity properties that we have when the measure μ is spread out, we are able to obtain some invariance properties for the limit measure of the potential kernel; we determine a set of periods for limit measures ν, that is the elements γ of the group Γ such that

$$\gamma * \nu = \nu,$$

and we show that the limit measure depends only on the speed of convergence to the boundary.

Theorem 3.4. *Suppose that the measure μ is spread out on Γ. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in Γ that converges to $\alpha \in \partial^*\mathbb{T}$ and such that the measures $g_n * U$ converge vaguely to ν. Then:*

1. *There exists a subsequence $\{g_{n_k}\}_k$ such that*
$$\lim_{k\to\infty} g_{n_k} g * U = \nu \qquad \text{for all } g \in \Gamma$$
2. *Every element of Γ that fixes α is a period for ν.*
3. *If $s \in \Gamma$ is such that $s\alpha = \alpha$ and $\phi(s) = 1$ then*
$$\lim_{n\to\infty} g_n * U = \lim_{n\to\infty} s^{\phi(g_n)} * U.$$

Proof. First suppose that $g_k = s^{n_k}$ with $s \in \Gamma$ such that $s\alpha = \alpha$ and $\phi(s) = 1$.

1. Possibly extracting a subsequence, one knows (Lemma 3.1) that the Radon measures

$$\nu_g = \lim_{k\to\infty} s^{n_k} g * U$$

are well defined for every $g \in \Gamma$ and that this family depends continuously on g. Therefore the group

$$P = \{b \in \mathrm{Hor}(\Gamma) : \nu_{bg} = \nu_g \,\forall g \in \Gamma\}$$

is closed. We first prove that P is normal in Γ. In fact, if $\gamma \in \Gamma$ and if the subsequence $\left\{s^{n'_k}\gamma s^{-n'_k}\right\}_k$ converges to γ' (see Lemma 3.2), then for all $b \in P$, using uniform left continuity,

$$\begin{aligned}\nu_{\gamma b\gamma^{-1}g} &= \lim_{k\to\infty} s^{n'_k}\gamma b\gamma^{-1} g * U = \lim_{k\to\infty} s^{n'_k}\gamma s^{-n'_k} * s^{n'_k} b\gamma^{-1} g * U =\\ &= \gamma' * \lim_{k\to\infty} s^{n'_k} b\gamma^{-1} g * U = \gamma' * \nu_{b\gamma^{-1}g} = \gamma' * \nu_{\gamma^{-1}g} = \gamma' * \gamma'^{-1} * \nu_g = \nu_g\end{aligned}$$

so that $\gamma b\gamma^{-1} \in P$.

Next we show that $P \setminus \mathrm{Hor}(\Gamma)$ is compact. In fact the preceding lemma says that for all $b \in \mathrm{Hor}(\Gamma)$ it is possible to find a sequence $\{r_k\}_k$ in $\mathrm{Hor}(\Gamma)$ that converges to an element $r \in K_\alpha$ and a subsequence such that $s^{n'_k} b^{-1} r_k s^{-n'_k}$ converges to the identity. Then, using uniform right continuity

$$\begin{aligned}\nu_{b^{-1}rg} &= \lim_{k\to\infty} s^{n'_k} b^{-1} r g * U = \lim_{k\to\infty} s^{n'_k} b^{-1} r_k g * U\\ &= \lim_{k\to\infty} s^{n'_k} b^{-1} r_k s^{-n'_k} * s^{n'_k} g * U = \lim_{k\to\infty} s^{n'_k} g * U = \nu_g\end{aligned}$$

i.e. $b^{-1}r \in P$, that is the class Pb has a representative r in K_α. Let now π be the projection of $\mathrm{Hor}(\Gamma)$ on $P \setminus \mathrm{Hor}(\Gamma)$. Then $P \setminus \mathrm{Hor}(\Gamma) = \pi(K_\alpha)$ is compact, because π is continuous and K_α is compact.

We will now show that $\nu_g = \nu$ for every $g \in \Gamma$. Fix a function $f \in C_c(\Gamma)$ and define the function h by

$$h(g) = \nu_g(f)$$

Note that h is bounded because the potential kernel is bounded, that it is continuous by uniform right continuity, and that it is μ-harmonic on the right:

$$\begin{aligned} h * \mu(g) &= \int_\Gamma \lim_{k\to\infty} s^{n_k} g\gamma * U(f)\mu(d\gamma) \\ &= \lim_{k\to\infty} \int_\Gamma s^{n_k} g\gamma * U(f)\mu(d\gamma) \text{ by dominated convercence} \\ &= \lim_{k\to\infty} s^{n_k} g * \sum_{i=0}^{\infty} \mu^{(i)} * \mu(f) \\ &= \lim_{k\to\infty} s^{n_k} g * \sum_{i=1}^{\infty} \mu^{(i)}(f) \\ &= \lim_{k\to\infty} s^{n_k} g * U(f) - f(s^{n_k} g) \\ &= h(g) \qquad \text{because } f \text{ has compact support.} \end{aligned}$$

As h is invariant by left translation of every element of the group P, it projects onto a function on $P \setminus \Gamma$,

$$\overline{h}(Pg) = h(g),$$

that is harmonic for the measure $\overline{\mu}$ obtained by projection of μ on $P \setminus \Gamma$. The group $P \setminus \Gamma$ is not Abelian, but taking the left quotient by the compact subgroup $P \setminus \mathrm{Hor}(\Gamma)$, one obtains

$$(P \setminus \mathrm{Hor}(\Gamma)) \setminus (P \setminus \Gamma) \cong \mathrm{Hor}(\Gamma) \setminus \Gamma \cong \mathbb{Z}.$$

Thus we are able to use a generalization to the compact extensions of $\mathbb{Z}$ of the Choquet-Deny theorem, due to Guivarc'h [8], Théorème V.2 (an aperiodic measure in that paper is a measure whose support generates Γ as closed group), and conclude that every continuous bounded harmonic function on $P \setminus \Gamma$ is constant. Hence the function $\overline{h}$ and therefore also h are constant and we can conclude that

$$\nu_g(f) = h(g) = h(e) = \nu_e(f) = \nu(f) \quad \forall g \in \Gamma.$$

2. We can now show that every $\gamma \in \Gamma$ that fixes α is a period of the limit measure ν. We first note that, as γ fixes α, the sequence $s^{-n_k}\gamma s^{n_k}$ is relatively compact because $s^{-n_k}\gamma s^{n_k}\alpha = \alpha$ and $\phi(s^{-n_k}\gamma s^{n_k}) = \phi(\gamma)$ (we have here a situation that is in some sense the opposite to Lemma 3.2, where we showed that $s^{n_k}\gamma s^{-n_k}$ is relatively compact). Let $\left\{s^{-n'_k}\gamma s^{n'_k}\right\}_k$ be a subsequence that converges to γ' then by right continuity

$$\gamma * \nu = \lim_{k\to\infty} s^{n'_k}\left(s^{-n'_k}\gamma s^{n'_k}\right) * U = \nu_{\gamma'} = \nu.$$

This ends the proof of points 1 and 2 when $g_k = s^{n_k}$.

3. Let now $\{g_k\}_k$ be a generic subsequence that converges to α. It can be decomposed in an horocyclic component and homothetic component by setting $g_k = b_k s^{n_k}$ with $b_k \in \mathrm{Hor}(\Gamma)$ and $n_k = \phi(g_k)$. We observe that $n_k \to +\infty$, so that we are able to apply to s^{n_k} what we have proved so far. On the other hand the sequence of the tree ends $\{b_k\alpha = g_k\alpha\}_k$ converges to α, thus $\{b_k\}_k$ is relatively compact and all its accumulation points are periods of the limit measure because they fix α. More precisely, for every subsequence along which $\{s^{n_k} g * U\}_k$ converges, we can extract

a sub-subsequence along which $\{b_{k_l}\}_l$ converges to a rotation r in K_α, then

$$\begin{aligned}\lim_{l\to\infty} g_{k_l} g * U &= \lim_{l\to\infty} b_{k_l} * s^{n_{k_l}} g * U \\ &= r * \lim_{l\to\infty} s^{n_{k_l}} g * U \\ &= \lim_{l\to\infty} s^{n_{k_l}} g * U \quad \text{because } r\alpha = \alpha\end{aligned}$$

Thus

$$\lim_{k\to\infty} g_k g * U = \lim_{k\to\infty} s^{n_k} g * U$$

and we conclude the proof of point 3.

The results of point 1 and 2 for a generic sequence $\{g_k\}_k$ are a straightforward consequence of the last equality and of the analogous results for $g_k = s^{n_k}$. □

When the group Γ acts on the tree in a sufficiently homogeneous and Abelian way, the results of the previous theorem hold even if we do not assume that μ is spread out. More precisely, we suppose that Γ satisfies the following hypotheses

(HA) There exists an end $\alpha_0 \in \partial^*\mathbb{T}$ such that the stabilizer, A, of α_0 in Γ is Abelian. Furthermore, there is a measurable set T of left coset representatives of A in Γ (that is, $\Gamma = TA$) and a reference homothety $c \in A$ with $\phi(c) = 1$ such that for every $t \in T$ we have

$$\lim_{n\to\infty} c^n t c^{-n} = e$$

In other words we require that Γ should be decomposed as product of an Abelian group of roto-homotheties A and of a set of translations T that act asymptotically like the identity far away from the center of the bottom boundary $\partial^*\mathbb{T}$. This hypothesis holds if Γ is contained in $\mathrm{Aff}(\mathbb{Q}_p) = \mathbb{Q}_p \rtimes \mathbb{Q}_p^*$ (in which case we can chose $T \subseteq \mathbb{Q}_p$ and $A \subseteq \mathbb{Q}_p^*$ and the contraction c may be chosen equal to $(0,p)$) and when Γ is the closure of the Lamplighter group (then T is the closure of $\mathcal{Z}_q$, and $A \cong \mathbb{Z}$).

Proposition 3.5. *Suppose that Γ satisfies (HA). Let $\{g_n\}_n$ be a sequence in Γ that converges to an end $\alpha \in \partial^*\mathbb{T}$ and such that $\{g_n * U\}_n$ converges to a limit measure ν. Then*

1. *There exists a subsequence $\{g_{n_k}\}_k$ such that*

$$\lim_{k\to\infty} g_{n_k} g * U = \nu \qquad \text{for all } g \in \Gamma$$

2. *Every element of Γ that fixes α is a period for ν.*
3. *If $s \in \Gamma$ is such that $s\alpha = \alpha$ and $\phi(s) = 1$ then*

$$\lim_{n\to\infty} g_n * U = \lim_{n\to\infty} s^{\phi(g_n)} * U. \tag{3.2}$$

Proof. The proof follows the same scheme of the previous theorem, being careful that now the potential kernel is not a priori uniformly right continuous.

Let be c the contraction that appears in the hypothesis (HA). First, we suppose that α is its center α_0, i.e. $c\alpha = \alpha$ and that $g_k = c^{n_k}$. For every function $f \in C_c(\Gamma)$ and every element $g = ta$ of the group Γ, the sequence $\{c^{n_k} g * U(f)\}_k$ is bounded and converges to $a * \nu(f)$. In fact for every convergent subsequence we have

$$\begin{aligned}\lim_{k\to\infty} c^{n'_k} g * U(f) &= \lim_{k\to\infty} c^{n'_k} t c^{-n'_k} c^{n'_k} a * U(f) \\ &= \lim_{k\to\infty} c^{n'_k} a * U(f) \quad \text{because of the left continuty} \\ &= a * \lim_{k\to\infty} c^{n'_k} * U(f) \quad \text{as } A \text{ is Abelian} \\ &= a * \nu(f)\end{aligned}$$

There fore it is possible to define the function h on Γ

$$h(g) = \lim_{k\to\infty} c^{n_k} g * U(f) = a * \nu(f),$$

which is continuous (because $a * \nu(f)$ is continuous), harmonic and bounded. As it projects to a continuous bounded function on the Abelian group A that is harmonic for the marginal of μ on A, it is has to be constant by the Choquet-Deny Theorem. This ends the proof of statement 1.

We have thereby also proved that

$$\nu(f) = a * \nu(f) \quad \text{for all } a \in A;$$

i.e. statement 2.

To prove statement 3 and to deal with a generic sequence $\{g_n\}_n$, we proceed exactly as in the proof of the previous theorem.

If the sequence $\{g_n\}_n$ converges to another end α, we just need to conjugate by an element of the group that maps α_0 to α. □

3.1.2. *Characterization.* This section is devoted to the characterization of the limit measures on a neighborhood of the bottom boundary $\partial^*\Gamma$. This characterization is given using the decomposition of Γ as semi-direct product of $\mathbb{Z}$ and $\mathrm{Hor}(\Gamma)$. Note that this decomposition depends on the choice of a reference homothety s and that the end $\alpha \in \partial^*\mathbb{T}$ such that $s\alpha = \alpha$ is then considered as the center of the bottom boundary of the tree. Also observe that we denote by $\widehat{\nu}$ the image of measure ν on a group under the map $g \mapsto g^{-1}$.

Theorem 3.6. *Suppose that* $\mathbb{E}[|\phi(X_1)|] < \infty$. *Then the following holds :*

1. If $\mu(\phi) = \mathbb{E}[\phi(X_1)] > 0$, *the only accumulation point of* $\{g * U\}_{g\in\Gamma}$ *when* g *converges towards a point of* $\partial^*\Gamma$ *is the zero measure.*

If we also suppose that μ *is spread out or that* Γ *satisfies (HA) then the following statements hold*

2. If $\mu(\phi) < 0$ *and* $\mathbb{E}[|X_1|] < +\infty$ *, then for every* $\beta \in \partial^*\Gamma$ *and for every* $b \in \Gamma$ *such that* $b\alpha = \beta$

$$\lim_{g\to\beta} g * U = b * m_{\langle s\rangle} * \widehat{\overline{m}}$$

where $m_{\langle s\rangle}$ *is the counting measure on the subgroup of* Γ *generated by* s *and* $\overline{m}$ *is the unique* $\hat{\mu}$*-invariant Radon measure on* $\mathrm{Hor}(\Gamma)$ *with total mass equal to* $-\frac{1}{\mu(\phi)}$ *that is invariant by right action of* K_α.

3. If $\mu(\phi) = 0$, $\mathbb{E}\big[\phi(X_1)^2\big] < +\infty$ *and* $\mathbb{E}\Big[|b(X_1)|^{2+\varepsilon}\Big] < +\infty$ *then for every* $\beta \in \partial^*\Gamma$ *and for every* $b \in \Gamma$ *such that* $b\alpha = \beta$

$$\lim_{g\to\beta} g * U = b * m_{\langle s\rangle} * \widehat{\overline{m}}$$

where $\overline{m}$ *is the unique* $\hat{\mu}$*-invariant Radon measure on* $\mathrm{Hor}(\Gamma)$ *defined as extension to* $\mathrm{Hor}(\Gamma)$ *of the* $\hat{\mu}$*-invariant measure (2.4) on* $\partial^*\mathbb{T}$.

Proof. 1. If $\mu(\phi) > 0$, the random walk $S_n = \phi(R_n)$ on $\mathbb{Z}$ is transient and the only accumulation point of its potential kernel , U_ϕ, in a neighborhood of $+\infty$ is zero (cf. proposition 3.4 [14]). For every bounded non negative function f with compact support on Γ there exists a bounded non-negative function F with finite support on $\mathbb{Z}$ such that $f(g) \le F(\phi(g))$; thus

$$0 \le \lim_{g\to\beta} g * U(f) \le \lim_{g\to\beta} g * U(F \circ \phi) = \lim_{g\to\beta} U_\phi(\phi(g), F) = 0$$

because $\phi(g)$ converges to $+\infty$ when g converges to $\beta \in \partial^*\mathbb{T}$.

2. and 3. Let $\{g_n\}_n$ be a sequence that converges to α and such that $\{g_n * U\}_n$ converges to a limit measure ν. We want to prove that $\hat{\nu} = m_{\langle s\rangle} * \widehat{\overline{m}}$.

Using Theorem 3.4, we can suppose that $g_n = s^{\phi_n}$ and we know that for all $g \in \Gamma$

$$\nu = \lim_{n\to\infty} s^{\phi_n} g * U.$$

Let $\widehat{U}$ be the potential measure associated with the measure $\hat{\mu}$, image of μ under group inversion. Then

$$\widehat{\nu} = \lim_{n\to\infty} \widehat{s^{\phi_n} g * U} = \lim_{n\to\infty} \widehat{U} * g^{-1} s^{-\phi_n}.$$

As $\{\phi_n\}_n$ converges to $+\infty$, for every function f with compact support in Γ, the functions

$$x \mapsto \left(f * s^{-\phi_n}\right)(x) = f(xs^{-\phi_n})$$

have their support in $\mathrm{Hor}(\Gamma) \times \mathbb{Z}_+$ for sufficiently large n. As $\hat{\mu}(\phi) = -\mu(\phi) \geq 0$, we can apply Corollary 2.8 to $\hat{\mu}$. Thus, there exists a probability $\overline{p}$ on $\mathrm{Hor}(\Gamma)$ such that for every sufficiently large n one has

$$\hat{U} * \overline{p} * s^{-\phi_n}(f) = \hat{U} * \overline{p}(f * s^{-\phi_n}) = \overline{m} * m_{\langle s \rangle}(f * s^{-\phi_n}) = \overline{m} * m_{\langle s \rangle}(f).$$

On the other hand, as $\overline{p}$ has finite mass, by dominated convergence

$$\lim_{n\to\infty} \hat{U} * \overline{p} * s^{-\phi_n}(f) = \int_\Gamma \lim_{n\to\infty} \hat{U} * g * s^{-\phi_n}(f)\overline{p}(dg) = \int_\Gamma \hat{\nu}(f)\overline{p}(dg) = \hat{\nu}(f).$$

This ends the proof in the case $\beta = \alpha$. When $\beta \neq \alpha$, one just need to multiply on the left by an element $b \in \Gamma$ such that $b\beta = \alpha$. □

3.2. The limit of the potential kernel at ω.

Theorem 3.7. *Suppose that μ is spread out and that $\phi(X_1)$ is integrable. If $\mu(\phi) < 0$, we also suppose that $\mathbb{E}[|X_1|] < +\infty$, while if $\mu(\phi) = 0$ we suppose that $\mathbb{E}\left[\phi(X_1)^2\right] < +\infty$ and $\mathbb{E}\left[|b(X_1)|^{2+\varepsilon}\right] < +\infty$. Then*

$$\lim_{g\to\omega} g * U = 0.$$

We would like to observe that as, it can be seen in the proof, the hypothesis that the measure is spread out is not needed when $\mu(\phi) \neq 0$ and g tends to ω in such a way that $\phi(g)$ goes to $+\infty$.

Proof. If g converges to ω in such a way that $\phi(g)$ is bounded from above we can directly apply the Theorem 2.16 in [6], which says that on every non-unimodular group, if the probability law of the random walk is spread out, the potential kernel converges to zero when g goes to infinity in such a way that the module of the Haar measure of the group (which is $\Delta(g) = q^{\phi(g)}$ in our case), is bounded from above.

Therefore we only need to show that for every sequence $\{g_n\}_n$ that converges to ω and such that $\{\phi(g_n)\}_n$ converges to $+\infty$ and for every non-negative continuous function f with compact support , the sequence $\{g_n * U(f)\}_n$ converges to zero. We will distinguish three cases according to sign of the drift $\mu(\phi)$.

Case 1: $\mu(\phi) > 0$. In this case one can directly apply the renewal theorem for the induced random walk on $\mathbb{Z}$, exactly as in the proof of Theorem 3.6.1 .

Case 2: $\mu(\phi) < 0$. First, note that in this case the renewal theorem on $\mathbb{Z}$ says that

$$\lim_{h\to+\infty} U_\phi(h,\cdot) = \frac{1}{-\mu(\phi)} m_{\mathbb{Z}}$$

where $m_{\mathbb{Z}}$ is the counting measure. On the other hand, we have just seen in Theorem 3.6 that if one identifies $\mathbb{Z}$ with the subgroup generated by the reference homothety s then

$$\lim_{h\to+\infty} s^h * U = m_{\mathbb{Z}} * \widehat{\overline{m}}$$

where $\widehat{\overline{m}}$ is a measure on $\mathrm{Hor}(\Gamma)$ whose mass is exactly $\frac{1}{-\mu(\phi)}$ (remember that the convolution $*$ takes place in Γ). Then for every compact set H in $\mathbb{Z}$ and every $\varepsilon > 0$

there exists a compact open set J_ε in $\mathrm{Hor}(\Gamma)$ such that

$$\begin{aligned}\lim_{h\to+\infty} s^h * U(HJ_\varepsilon^c) &= \lim_{h\to+\infty} s^h * U(H\mathrm{Hor}(\Gamma) - HJ_\varepsilon)\\ &= \lim_{h\to+\infty} U_\phi(h,H) - \lim_{h\to+\infty} s^h * U(HJ_\varepsilon)\\ &= m_{\mathbb{Z}}(H)(\frac{1}{-\mu(\phi)} - \widehat{\widehat{m}}(J_\varepsilon)) < \varepsilon;\end{aligned}$$

i.e. the family of measures $\left\{s^h * U(H\cdot)\right\}_{h\in\mathbb{N}}$ is tight on $\mathrm{Hor}(\Gamma)$. Now fix a compact set K in Γ and observe that $g = b(g)s^{\phi(g)}$. Then

$$g * U(K) = U(g^{-1}K) = U(s^{-\phi(g)}b(g)^{-1}K) = s^{\phi(g)} * U(b(g)^{-1}K).$$

Note that for every sequence $\{g_n\}_n$ that converges to ω in such a way that $\{\phi(g_n)\}_n$ converges to $+\infty$, also its projection $\{b(g_n)\}_n$ on the horocyclic group and its inverse $\left\{b(g_n)^{-1}\right\}_n$ converge to ω. Let H be a compact set of $\mathbb{Z}$ such that $\phi(K) \subseteq H$, then for every $\varepsilon > 0$ and for all $x \in K$

$$b(g_n)^{-1}x = s^{\phi(x)}s^{-\phi(x)}b(g_n)^{-1}s^{\phi(x)}b(x) \in HJ_\varepsilon^c \qquad \text{for any sufficiently large } n\ .$$

Thus, for sufficiently large n, we have $b(g_n)^{-1}K \subseteq HJ_\varepsilon^c$. We can conclude that

$$\overline{\lim_{n\to\infty}} g_n * U(K) = \overline{\lim_{n\to\infty}} s^{\phi(g_n)} * U(b(g_n)^{-1}K) \le \overline{\lim_{n\to\infty}} s^{\phi(g_n)} * U(HJ_\varepsilon^c) < \varepsilon$$

i.e. $g_n * U(K)$ converges to zero.

Case 3: $\mu(\phi) = 0$. Let $m = \max\{\phi(g) : g \in \operatorname{supp} f\}$ and for every fixed $g \in \Gamma$ let t be the first time when $\phi(g) + S_n = \phi(gR_n)$ is below m

$$t = \inf\{k \ge 0 : \phi(g) + S_k \le m\}.$$

Thus for every $n < t$ on has $f(gR_n) = 0$, and therefore

$$g * U(f) = \mathbb{E}\left[\sum_{n=0}^{+\infty} f(gR_n)\right] = \mathbb{E}\left[\sum_{n=0}^{t-1} f(gR_n)\right] + \mathbb{E}[gR_t * U(f)] = \mathbb{E}[gR_t * U(f)]\,.$$

We have already seen that, when γ converges to ω in such a way that $\phi(\gamma)$ is bounded from above, $\gamma * U(f)$ converges to 0. Hence for every $\varepsilon > 0$, there is a compact set K_ε in Γ such that

$$\gamma * U(f) \le \varepsilon \qquad \text{for all } \ \gamma \in K_\varepsilon^c \cap [\phi \le m].$$

By definition $\phi(gR_t) \le m$ therefore

$$\begin{aligned} g * U(f) &= \mathbb{E}\big[(gR_t * U(f))1_{[\phi(gR_t)\le m]}\big]\\ &\le \varepsilon + \mathbb{E}\big[(gR_t * U(f))1_{[gR_t\in K_\varepsilon]}\big]\\ &\le \varepsilon + C\mathbb{E}\big[1_{[gR_t\in K_\varepsilon]}\big]\end{aligned}$$

where C is the upper bound of the kernel $g * U(f)$. Let now $\left\{l_k^-\right\}_k$ the sequence of the ladder stopping times when $S_n = \phi(R_n)$ reaches its minima:

$$l_k^- = \min\{n > l_{k-1}^- :\ S_n < S_{l_{k-1}^-})\} \ \text{ et } \ l_0^- = 0.$$

Because $\phi(R_t)$ is strictly smaller than the minimum of $\phi(R_n)$ for $n < t$, there exists $i \in \mathbb{N}$ such that $t = l_i^-$. Let U_{l^-} be the potential measure of the random walk $\left\{R_{l_k^-}\right\}_{k\in\mathbb{N}}$

$$\begin{aligned} g * U(f) &\le \varepsilon + C\mathbb{E}\big[1_{[gR_t\in K_\varepsilon]}\big] = \varepsilon + C\mathbb{E}\left[1_{[gR_{l_i^-}\in K_\varepsilon]}\right]\\ &\le \varepsilon + C\mathbb{E}\left[\sum_{k=0}^{\infty} 1_{[gR_{l_k^-}\in K_\varepsilon]}\right] = \varepsilon + C\, g * U_{l^-}(K_\varepsilon)\end{aligned}$$

We remark that the random walk $R_{l_k^-}$ satisfies the hypothesis needed to apply Case 2. Indeed, $\mathbb{E}\Big[\phi(R_{l_1^-})\Big] < 0$, and Proposition 4 in [4] states that the norm of $R_{l_1^-}$ is integrable, while Lemma 2.26 in [6] guarantees that its law is spread out (the hypothesis that the group is almost connected that is assumed there is not necessary to prove this result). Thus

$$\overline{\lim_{g \to \omega}} g * U(f) \leq \varepsilon + \overline{\lim_{g \to \omega}} g * U_{l^-}(K_\varepsilon) = \varepsilon;$$

for every positive ε and we can conclude. □

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