ON UNBOUNDED INVARIANT MEASURES OF STOCHASTIC DYNAMICAL SYSTEMS

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Abstract. We consider stochastic dynamical systems on $\mathbb{R}$, i.e. random processes defined by $X_n^x = \Psi_n(X_{n-1}^x)$, $X_0^x = x$, where Ψ_n are i.i.d. random continuous transformations of some unbounded closed subset of $\mathbb{R}$. We assume here that Ψ_n behaves asymptotically like $A_n x$, for some random positive number A_n (the main example is the affine stochastic recursion $\Psi_n(x) = A_n x + B_n$). Our aim is to describe invariant Radon measures of the process X_n^x in the critical case, when $\mathbb{E}\log A_1 = 0$. We prove that those measures behave at infinity like $\frac{dx}{x}$. We study also the problem of uniqueness of the invariant measure. We improve previous results known for the affine recursions and generalize them to a larger class of stochastic dynamical systems which includes, for instance, reflected random walks, stochastic dynamical systems on the unit interval $[0,1]$, additive Markov processes and a variant of the Galton-Watson process.

1. Introduction

1.1. **Stochastic dynamical systems.** Let $\mathfrak{F}$ be the semigroup of continuous transformations of an unbounded closed subset $\mathcal{R}$ of the real line $\mathbb{R}$ endowed with the topology of uniform convergence on compact sets. In the most interesting examples $\mathcal{R}$ is the real line, the half-line $[0,+\infty)$ or the set of natural numbers $\mathbb{N}$. Given a regular probability measure μ on $\mathfrak{F}$, we define the stochastic dynamical system (SDS) on $\mathcal{R}$ by

$$\begin{aligned} X_0^x &= x; \\ X_n^x &= \Psi_n(X_{n-1}^x), \end{aligned} \tag{1.1}$$

where $\{\Psi_n\}$ is a sequence of i.i.d. random functions, distributed according to μ.

The aim of this paper is to study conditions for the existence and uniqueness, as well as behavior at infinity, of an invariant infinite Radon measure of the process X_n^x, i.e. of a measure ν on $\mathbb{R}$ such that

$$\mu *_{\mathfrak{F}} \nu(f) = \nu(f), \tag{1.2}$$

for any $f \in C_C(\mathbb{R})$, where

$$\mu *_{\mathfrak{F}} \nu(f) = \int_{\mathbb{R}} \mathbb{E}\big[f(X_1^x)\big]\nu(dx) = \int_{\mathfrak{F}}\int_{\mathbb{R}} f(\Psi(x))\nu(dx)\mu(d\Psi).$$

There is quite an extensive literature on the case when the process X_n is positive recurrent, that is, it possesses an invariant *probability* measure. The existence of such a measure can be proved supposing that the process has some contractive property (for example, if Ψ_n are Lipschitz mappings with Lipschitz coefficients $L_n = L(\Psi_n)$ and $\mathbb{E}[\log L_1] < 0$), [11]). This invariant probability measure

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is well described in several specific cases, such as affine recursions (i.e. $\Psi(x) = Ax + B$), namely in the seminal paper of H.Kesten [17]. C.M.Goldie [15] and recently M. Mirek [21] generalized Kesten's theorem to stochastic recursions such that $\Psi(x)$ behaves like Ax for large x. They proved that if $\mathbb{E}A^\kappa = 1$ (and some other hypotheses are satisfied), then

$$\lim_{z\to\infty} z^\kappa \nu\{x : |x| > z\} = C_+ > 0.$$

In other words the measure ν is close at infinity to $\frac{C_+ dx}{x^{1+\kappa}}$.

Less is known for the null recurrent case, especially in a general setting. Existence and uniqueness of an invariant Radon measure have been the topic of two recent works: B. Deroin, V. Kleptsyn, A. Navas and K. Parwani [10] on symmetric SDS of homeomorphism of $\mathbb{R}$, and M. Peigné and W. Woess [22] on the phenomenon of local contraction. We refer to them for a more complete bibliography on the subject. As in the contracting case, affine recursions is one of the first models being systematically approached. A seminal paper in this area is the one of Babillot, Bougerol and Elie [2]. They proved existence and uniqueness of a Radon measure and gave a first result on its behavior at infinity.

The goal of the present work is twofold. First of all we investigate the behavior at infinity of invariant measures and, for a large class of SDS's, we generalize and improve results known for affine recursions. Secondly, we consider the problem of uniqueness of the invariant measure. We give a relatively simple criterium that can be applied for very concrete examples.

1.2. **Behavior at infinity.** It turns out that to prove existence and to describe the tail of the measure it is sufficient to control the maps that generate the SDS near infinity. In particular we suppose that they are asymptotically linear, in the sense that there exists $0 \le \alpha < 1$ such that for all $\psi \in \mathfrak{F}$

$$(AL^\alpha) \qquad |\psi(x) - A_\alpha(\psi)x| \le B_\alpha(\psi)(1 + |x|^\alpha) \qquad \text{for all } x \in \mathcal{R}$$

with $A_\alpha(\psi)$ and $B_\alpha(\psi)$ strictly positive. We study here the critical case, i.e. $\mathbb{E}[\log A_\alpha] = 0$.

Existence of an invariant measure supported in $\mathcal{R}$ is relatively easy to deduce from the well-known literature, because in this case the SDS is bounded by a recurrent process (we give more details in subsection 2.3). The main result of the paper is the description of the tail of invariant measures at infinity.

Theorem 1.3. *Suppose that there exists $0 \le \alpha < 1$ such that the maps Ψ_n satisfy (AL^α) μ-a.s. and that*

(1.4) $\qquad \mathbb{E}[\log A_\alpha] = 0$ *and* $\mathbb{P}[A_\alpha = 1] < 1,$

(1.5) $\qquad \mathbb{E}\big[(|\log A_\alpha| + \log^+ |B_\alpha|)^{2+\varepsilon}\big] < \infty,$

(1.6) $\qquad$ *the law of* $\log A_\alpha$ *is aperiodic, i.e. there is no* $p \in \mathbb{R}$ *such that* $\log A_\alpha \in p\mathbb{Z}$ *a.s.*

Let ν be an invariant Radon measure ν for the process $\{X_n^x\}_n$. Suppose that ν is supported by $\mathcal{R}$ and it is positive on any neighborhood of $+\infty$. Then the family of dilated measures $\delta_{z^{-1}} * \nu(I) := \nu(zI)$ converges vaguely on $\mathbb{R}_+^* = (0,\infty)$ to $C_+\frac{da}{a}$ as z goes to infinity for some $C_+ > 0$, i.e.

$$\lim_{z\to\infty} \int_{\mathbb{R}_+^*} \phi(z^{-1}u)\nu(du) = C_+ \int_{\mathbb{R}_+^*} \phi(a)\frac{da}{a},$$

for any $\phi \in C_C(\mathbb{R}_+^*)$.

The key example of an asymptotic linear SDS is the affine recursion (called also the random difference equation). Then $\mathfrak{F}$ is the set of affine mappings of the real line $\Psi(x) = Ax + B$ with $A > 0$ and the process is given by the following formula

$$X_n^x = A_n X_{n-1}^x + B_n, \qquad X_0^x = x. \tag{1.7}$$

Our results are also valid for Goldie's recursions, e.g. $\Psi(x) = \max\{Ax, B\} + C$ (with $A > 0$) and $\Psi(x) = \sqrt{A^2x^2 + Bx + C}$ (with A, B, C positive). Since the problem can be reduced, without any loss of generality, to the case $\alpha = 0$ (see Lemma 2.1), our hypotheses essentially coincide, in the one dimensional situation, with the class introduced by M.Mirek [21]. Our main theorem should be viewed as an analog of Kesten's and Goldie's results in the critical case.

Other interesting examples can be obtained conjugating asymptotic linear systems with an appropriate homeomorphism. For instance, our result can also be applied to describe invariant measures of SDS on the interval generated by functions that have the same derivative at the two extremities. Theorem 1.3 also says that invariant measures of SDS on $[0, +\infty)$ generated by mappings exponentially asymptotic to translations, i.e.

$$|\psi(x) - x + u_\psi| \le v_\psi e^{-x}, \qquad \forall x \ge 0$$

behave at infinity as the Lebesgue measure dx of $\mathbb{R}$, if $\mathbb{E}(u_\phi) = 0$. This result can be compared with the Choquet-Deny Theorem saying that the only invariant measure for centered random walks on $\mathbb{R}$ is the Lebesgue measure. Another interesting process that is α-asymptotically linear for $\alpha > 1/2$ is a Galton-Watson evolution process with random reproduction laws. In Section 6, we give more details on the different examples.

Let us mention that in our previous papers [3, 6, 7] we have already studied the behavior at infinity of the invariant measure ν for the random difference equation (1.7). However the main results were obtained there under much stronger assumptions, namely we assumed existence of exponential moments, that is $\mathbb{E}[A^\delta + A^{-\delta} + |B|^\delta] < \infty$ for some $\delta > 0$. Theorem 1.3 improves all our previous results for affine recursions and describes the asymptotic behavior of ν under optimal assumptions, that is the weakest known conditions implying existence of the invariant measure [2]. To our knowledge, for all the other recursions even partial results are not known.

We would like also to remark that, in the contracting case, Kesten's theorem requires moment of order at least κ and, as far as we know, there exist no results on the behavior of the tail of the invariant probability when the measure is supposed to have only logarithmic moment.

The proof of Theorem 1.3 is given in sections 3 and 4. In order to describe ν at infinity, we give first an upper bound of this measure and prove some regularity properties of its quotient. The techniques we use in the present paper are more powerful than those applied in [6], and are heavily based on the renewal theory for random walks on the affine group. Among other results we prove directly that $\nu[-z, z]$ grows as $\log z$ (Proposition 3.1). Next, in Section 4 we consider the Poisson equation for the additive convolution on $\mathbb{R}$

$$f(x) = \overline{\mu} * f(x) + g(x),$$

where $f(x) = \int \phi(e^{-x}u)\nu(du)$ for some $\phi \in C_C(\mathbb{R}_+^*)$ and $\overline{\mu}$ is the law of $-\log A$. Notice that the asymptotic behavior of f and ν is the same, therefore it is sufficient to study f. In the contrast to [6] we do not solve explicitly this equation. We apply techniques borrowed from the work of Durrett and Liggett [12] (see also Kolesko [18]), reduce the problem to the classical renewal equation with drift and deduce its asymptotic behavior from the renewal theorem.

1.3. **Uniqueness of the invariant measure.** Another fundamental question is to determine whether the invariant measure is unique or not. The nature of this problem is different from the ones we have considered so far. In fact dniqueness depends on the local behavior of the system and it is no more sufficient to control the random maps only at the infinity.

In the non-contracting case, this problem was studied first by Babillot, Bougerol and Elie [2] in the context of the affine recursion and they proved uniqueness under the assumptions of Theorem 1.3. Relying on their ideas Benda [3] studied in full generality recurrent and locally contractive SDS's.

The SDS is called recurrent if there exists a closed set L such that every open set intersecting L is visited by X_n^x infinitely often with probability 1. The SDS is locally contractive if for any $x, y \in \mathbb{R}$ and every compact set $K \subset \mathbb{R}$,

$$\lim_{n\to\infty} |X_n^x - X_n^y| \cdot \mathbf{1}_K(X_n^x) = 0 \qquad \text{almost surely.} \tag{1.8}$$

Benda [3] proved that if $\{X_n^x\}$ is a recurrent and locally contractive SDS, then it possesses a unique (up to a multiplicative constant) invariant Radon measure. He didn't publish his results, however they have been recently incorporated, with a complete and simplified proof, into two papers of Peigné and Woess [22, 23], where they also investigated ergodicity of SDS generated by Lipschitz maps with centered Lipschitz's coefficient.

Our aim is to consider very concrete families of Lipschitz mappings of $\mathbb{R}_+$, as the one presented in Goldie's work [15]. Although recurrence of the corresponding SDS's is immediate, the main obstacle in applying Benda's theorem is the local contraction hypothesis (1.8). In [23] the authors considered the reflected affine stochastic recursion, being a mixture of the reflected random walk (described below) and the affine stochastic recursion (defined in 1.7). Unfortunately the method of hyperbolic extensions they introduce cannot be applied to dynamical systems, whose dependence on the affine recursion cannot be expressed in such a direct way.

A different approach can be found in [10], where the authors proved a local contraction property for a symmetric SDS generated by homeomorphisms of $\mathbb{R}$. Their proof is very elegant but is heavily based on the additional assumption that the SDS is generated by *invertible* mappings distributed according to a *symmetric* measure. In particular their results cannot be applied to noninvertible SDS, as the one generated by $\psi(x) = \max\{Ax, B\} + C$, one of the most interesting in applications.

Our contribution to the subject is to give sufficient conditions for uniqueness that can be applied to some concrete mappings of $\mathbb{R}_+ = [0, \infty)$, such as $\psi(x) = \max\{Ax, B\} + C$ and other Goldie's recursions.

Theorem 1.9. *Suppose that $\mathcal{R} = [0, \infty)$, $\alpha = 0$ and that the hypotheses of Theorem 1.3 are satisfied. Assume moreover that*

(1) there exists $\beta > 0$ such that $\mathbb{P}(\Psi[0, +\infty) \subseteq [\beta, +\infty)) > 0$;
(2) $A(\Psi)x \le \Psi(x) \le A(\Psi)x + B(\Psi)$ for all $x \ge 0$;
(3) the functions Ψ are Lipschitz and their Lipschitz coefficients are equal to $A(\Psi)$.

Then the SDS defined on $[0, \infty)$ by (1.1) is locally contractive. Therefore there exists a unique invariant Radon measure of the process $\{X_n^x\}$ on $[0, +\infty)$.

The proof of this theorem is contained in Section 5.

1.4. **Reflected random walk.** The reflected random walk is the SDS defined for $x \in \mathbb{R}_+ = [0, \infty)$, by

$$\begin{aligned} Y_0^x &= x, \\ Y_n^x &= \left|Y_{n-1}^x - u_n\right|, \end{aligned} \tag{1.10}$$

where u_n is a sequence of i.i.d. real valued random variables with a given law μ.

If $u_n \ge 0$ a.s., then it was proved by Feller [14] that this process possesses a unique invariant probability measure ν, i.e. a measure satisfying

$$\mu * \nu(f) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} f(|x - y|)\nu(dx)\mu(dy) = \int_{\mathbb{R}_+} f(x)\nu(dx) = \nu(f).$$

Moreover the measure ν can be explicitly computed: $\nu(dx) = (1 - F(x))dx$, for F being the distribution function of μ. The process has been also studied in more general settings, when u_n admits also negative values (see Peigné, Woess [22] for recent results and a comprehensive bibliography).

Here, we are interested in the critical case when $\mathbb{E}u_n = 0$. Peigné and Woess [22] proved that if $\mathbb{E}(u_1^+)^{\frac{3}{2}} < \infty$, for $u_1^+ = \max\{u_1, 0\}$, then the process $\{X_n\}$ is recurrent on $\mathbb{R}_+$. As a consequence of Benda's theorem, the process possesses a unique invariant Radon measure ν on $\mathbb{R}_+$ (local contractivity is easy to prove). The reflected random walk can be transformed in an asymptotically linear system by conjugating with an invertible function s of $[0, +\infty)$ such that $s(x) = \mathrm{e}^x$ for large x. Then $\psi(x) = s(|s^{-1}(x) - u|)$ is asymptotically linear with $A(\psi) = e^{-u}$. Hence Theorem 1.3 can be used to justify that the invariant measure of Y_n^x behaves at infinity like the Lebesgue measure. Nevertheless in this case one can prove the same result under weaker moment assumptions and a much simpler proof. A short argument based only on the duality lemma and the renewal theorem gives:

Theorem 1.11. *Assume $\mathbb{E}u_1 = 0$, $\mathbb{E}(u_1^+)^{\frac{3}{2}} < \infty$, $\mathbb{E}(u_1^-)^2 < \infty$ and the law μ of u_1 is aperiodic, then for every $\phi \in C_C(\mathbb{R}_+)$*

$$\lim_{x\to\infty} \int_{\mathbb{R}_+} \phi(u - x)\nu(du) = C_+ \int_{\mathbb{R}_+} \phi(u)du,$$

for some positive constant C_+.

The proof of this theorem will be given in Section 6.6.

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2. Notations and preliminary results

2.1. **Reduction to condition (AL).** Observe first that, conjugating the SDS with an appropriate function, we can suppose without loss of generality that the distance of the random map to a linear function is smaller than some constant. In fact we have the following lemma whose proof is postponed to Appendix A:

Lemma 2.1. *Let $0 \le \alpha < 1$. Suppose that ψ satisfies*

$$(AL^\alpha) \qquad |\psi(x) - A_\alpha x| \le B_\alpha(1 + |x|^\alpha).$$

Then the conjugate function $\psi_r = r \circ \psi \circ r^{-1}$, where $r(x) = \mathrm{sign}(x)|x|^{1-\alpha}$, satisfies (AL^0) with $A_0 = A_\alpha^{1-\alpha}$. The appropriate constant B_0 can be chosen such that $\log^+ B_0 \le C_\alpha(|\log A_\alpha| + \log^+ B_\alpha + 1)$, for the constant C_α depending only on α.

If ψ is distributed according to μ, the law ψ_r is given by $\mu_r = \delta_r * \mu * \delta_{r^{-1}}$, and if ν is a μ-invariant measure then $\nu_r = \delta_r * \nu$ is μ_r-invariant. Thus if Theorem 1.3 holds for ν_r then it holds for ν. Indeed

$$\begin{aligned}
\lim_{z\to\infty} \int_{\mathbb{R}_+^*} \phi(z^{-1}u)\nu(du) &= \lim_{z\to\infty} \int_{\mathbb{R}_+^*} \phi(z^{-1}r^{-1}(u))\nu_r(du) = \lim_{z\to\infty} \int_{\mathbb{R}_+^*} \phi(z^{-1}u^{1/(1-\alpha)})\nu_r(du) \\
&= \lim_{z\to\infty} \int_{\mathbb{R}_+^*} \phi((z^{-(1-\alpha)}u)^{1/(1-\alpha)})\nu_r(du) = C_+ \int_{\mathbb{R}_+^*} \phi(a^{1/(1-\alpha)}) \frac{da}{a} \\
&= C_+(1-\alpha) \int_{\mathbb{R}_+^*} \phi(a) \frac{da}{a}.
\end{aligned}$$

In order to simplify our notations, we'll suppose from now on that $\alpha = 0$, i.e. for all $\psi \in \mathfrak{F}$

$$(AL) \qquad A(\psi)x - B(\psi) < \psi(x) < A(\psi)x + B(\psi) \qquad \text{for all } x \in \mathcal{R}.$$

Since $\mathcal{R}$ is closed, we can extend the property (AL) to all $x \in \mathbb{R}$ for a suitable continuous extension of ψ to $\mathbb{R}$. With a slight abuse of notation, we will denote with the same letter (e.g. ψ), the map from $\mathcal{R}$ to $\mathcal{R}$ and its continuous extension that verifies (AL) for all $x \in \mathbb{R}$. In the same way, ν will be seen both as a measure on $\mathcal{R}$ and as a measure on $\mathbb{R}$ whose support is contained in $\mathcal{R}$.

2.2. **Comparison of X_n^x with the affine recursion.** We assume that the maps $A = A(\psi)$ and $B = B(\psi)$ from $\mathfrak{F}$ to $\mathbb{R}_+^* = (0, \infty)$ are measurable and that $\mathfrak{F}$ is a monoid closed by composition. Assumption (AL) implies

$$\lim_{\substack{x \to +\infty \\ x \in \mathcal{R}}} \psi(x)/x = \lim_{\substack{x \to -\infty \\ x \in \mathcal{R}}} \psi(x)/x = A(\psi),$$

therefore the map A is a homomorphism from $\mathfrak{F}$ to $\mathbb{R}_+^*$ i.e. $A(\psi_1 \circ \psi_2) = A(\psi_1)A(\psi_2)$. The choice of B is not unique and it can be chosen as big as needed.

Let $\{\Psi_n\}_{n=1}^\infty$ be an i.i.d. sequence of random variables with values in $\mathfrak{F}$ of law μ. We are interested in the study of the iterated stochastic function system

$$X_n^x = \Psi_n(X_{n-1}^x) = \Psi_n ... \Psi_1(x) \text{ and } X_0^x = x.$$

If hypothesis (AL) is satisfied, the trajectories of the process X_n^x can be dominated from below and from above by the affine recursions

$$Z_n^x = A_n Z_{n-1}^x - B_n \text{ and } Y_n^x = A_n Y_{n-1}^x + B_n, \tag{2.2}$$

where, to simplify our notation, we note $A_n = A(\Psi_n)$ and $B_n = B(\Psi_n)$. We will also assume, according to hypotheses of Theorem 1.3, a logarithmic moment of order $2 + \varepsilon$ and that $\log A_1$ is nontrivially centered. Without any loss of generality, we can also choose $B(\psi)$, such that:

$$B_n \geq 1 \text{ a.s.} \tag{2.3}$$

$$\mathbb{P}(A_n x + B_n = x) < 1 \text{ for all } x. \tag{2.4}$$

In such a way the two dimensional process (Z_n^x, Y_n^x) satisfies all the assumptions required by Babillot, Bougerol, Elie [2]. Thus it is recurrent, locally contractive and possesses a unique invariant measure.

It will be convenient to use in the proof the language of groups. Namely, let $G = \text{Aff}(\mathbb{R}) = \mathbb{R} \rtimes \mathbb{R}_+^*$ be the group of all affine mappings of $\mathbb{R}$, i.e. the set of pairs $(b, a) \in \mathbb{R} \times \mathbb{R}_+^*$ acting on $\mathbb{R}$: $(b, a) : x \mapsto ax + b$. Then the group product is given by the formula

$$(b, a) \cdot (b', a') = (b + ab', aa'),$$

the identity element is $(0, 1)$ and the inverse element is given by

$$(b, a)^{-1} = (-b/a, 1/a).$$

Let μ_G be the probability distribution of (B_n, A_n) on the group G. Then the random elements $g_n = (B_n, A_n)$ are i.i.d. random variables in G with law μ_G. We define the left and the right random walk on G:

$$L_n = g_n \cdot ... \cdot g_1, \qquad R_n = g_1 \cdot ... \cdot g_n. \tag{2.5}$$

Then, $Y_n^x = L_n(x)$.

A very important role in our proofs will be played by the random walk on $\mathbb{R}$ generated by $-\log A_i$, i.e.

$$S_n = -(\log A_1 + \cdots + \log A_n), \tag{2.6}$$

(we put the sign minus to follow notations of our previous works). Since $\mathbb{E} \log A = 0$, the random walk S_n is recurrent. Moreover since we assume aperiodicity, the support of S_n is just $\mathbb{R}$. We often use the downward and upward sequence of stopping times

$$l_n := \inf\{k > l_{n-1} : S_k < S_{l_{n-1}}\}, \quad t_n := \inf\{k > t_{n-1} : S_k \geq S_{t_{n-1}}\} \tag{2.7}$$

and $l_0 = t_0 = 0$. Observe that t_1 and l_1 are almost surely finite, but have infinite mean. On the other hand, hypothesis $\mathbb{E}(|\log A|^{2+\varepsilon}) < \infty$ guarantees that S_{t_1} and S_{l_1} are integrable (see [9]).

In the sequel we will use, depending on the situation, different convolutions. We define a convolution of a function f on $\mathbb{R}$ with a measure η on $\mathbb{R}$ as a measure on $\mathbb{R}$ given by

$$f * \eta(K) = \int_{\mathbb{R}} \mathbf{1}_K(f(u))\eta(du) = \eta(f^{-1}(K)). \tag{2.8}$$

Given $z \in \mathbb{R}_+^*$ and a measure η on $\mathbb{R}$ we define

$$\delta_z *_{\mathbb{R}_+^*} \eta(K) = \int_{\mathbb{R}} \mathbf{1}_K(zu)\eta(du) = \eta(z^{-1}K). \tag{2.9}$$

2.3. **Existence of an invariant measure.** We conclude this section observing that the existence of the invariant measure on $\mathcal{R} \subseteq \mathbb{R}$ for a SDS satisfying the hypotheses of Theorem 1.3 follows immediately from recurrence of the process $\{X_n^x\}$ and Lin's theorem [20].

More precisely, consider the positive operator $Pf(x) = \int f(\Psi(x))\mu(d\Psi)$ on $C_b(\mathcal{R})$. Then, since $Z_n^x \le X_n^x \le Y_n^x$ and (Z_n^x, Y_n^x) is recurrent, the process $\{X_n^x\}$ is recurrent, i.e. there exists a nonnegative function $u \in C_c(\mathbb{R})$ such that $\sum_{n=0}^\infty P^n u(x) = \infty$ for all x. Therefore, by [20] there exists a non-null invariant Radon measure ν on $\mathcal{R}$ of the process $\{X_n^x\}$.

Observe that the support of this measure can be bounded (for instance if the functions Ψ fix the point 0, then the Dirac measure at 0 is an invariant measure). In this paper we are interested in measures having unbounded support. A sufficient (but not necessary) condition to ensure that the invariant measure is not bounded is to assume that the random functions Ψ do not fix a compact subset C of $\mathbb{R}$ (that is there is no compact C such that $\mathbb{P}(\Psi(C) \subseteq C) = 1$).

3. First bounds of the tail of the invariant measure

We start to study the behavior of ν at infinity. In particular we will prove in this section that $\nu(dx)$ does not grow faster than $\frac{dx}{x}$, the Haar measure of $\mathbb{R}_+^*$. The behavior of ν at ∞ is related to the behavior of the family of measures $\delta_{z^{-1}} * \nu$. In this section we prove

Proposition 3.1. *Under the hypotheses of Theorem 1.3 we have the following:*

(1) There exists $C_0 > 0$ such that

$$\nu[-z, z] < C_0(1 + \log z) \quad \text{for all } z > 1$$

Moreover, if the support of ν is not bounded on the right, i.e. $\nu(z, +\infty) > 0$ for all $z \in \mathbb{R}$, then

(2) There exist $M > 1$ and $\delta > 0$ such that $\nu[z, zM] > \delta$ for all $z \ge 1$.

(3) For all $u_2 > u_1 > 0$ there exists $C = C(u_1, u_2, M) > 0$ such that

$$\frac{\nu[\mathrm{e}^{x+y}u_1, \mathrm{e}^{x+y}u_2]}{\nu[\mathrm{e}^x, \mathrm{e}^x M]} < C(1 + y) \quad \text{for all } x > 0, y > 0. \tag{3.2}$$

In particular the family of measures $\frac{1}{\nu[z,zM]}\delta_{z^{-1}} * \nu$ on $(0, +\infty)$ is vaguely compact when z goes to $+\infty$.

There are two key arguments in the proof of this proposition. One is the following Lemma, that we will use several times in the sequel.

Lemma 3.3. *Let ν be a positive μ-invariant measure on $\mathbb{R}$. Then for any pair of intervals $V, U \subset \mathbb{R}$,*

$$\nu(V) \ge \mathbb{P}(T_{\mathfrak{W}} < \infty) \cdot \nu(U)$$

where

$$\mathfrak{W} = \mathfrak{W}(V, U) = \{\psi \in \mathfrak{F} \mid \psi(U) \subset V\}$$

and $T_{\mathfrak{W}}$ is the stopping time defined by $T_{\mathfrak{W}} = \inf\{n \ge 0 : \Psi_1 \cdots \Psi_n \in \mathfrak{W}\}$.

Proof. Observe that the backward process

$$M_n = \Psi_1 \cdots \Psi_n * \nu(V) \qquad M_0 = \nu(V)$$

is a positive martingale with respect to the filtration generated by the Ψ_n. In fact

$$\mathbb{E}(M_n|\mathcal{F}_{n-1}) = \Psi_1 \cdots \Psi_{n-1} * \mu * \nu(V) = \Psi_1 \cdots \Psi_{n-1} * \nu(V).$$

Since $(\Psi_1 \cdots \Psi_{T_{\mathfrak{W}}})^{-1}(V) \supseteq U$, for any fixed $n \in \mathbb{N}$, by the optional stopping time theorem,

$$\nu(V) = \mathbb{E}(M_{T_{\mathfrak{W}} \wedge n}) \geq \mathbb{E}(\mathbf{1}_{\{T_{\mathfrak{W}} \leq n\}} \Psi_1 \cdots \Psi_{T_{\mathfrak{W}}} * \nu(V)) \geq \mathbb{P}(T_{\mathfrak{W}} < n)\nu(U).$$

We let n go to infinity to conclude. □

The other crucial observation is that the backward recursion $\Psi_1 \cdots \Psi_n(x)$ is controlled by the right random walk R_n on the affine group generated by the product of $g_i = (B_i, A_i)$ (see (2.5)). More precisely, given $g \in \mathrm{Aff}(\mathbb{R})$ we denote by $a(g)$ and $b(g)$ its projections on $\mathbb{R}_+^*$ and $\mathbb{R}$ respectively, then

$$a(R_n)x - b(R_n) \leq \Psi_1 \cdots \Psi_n(x) \leq a(R_n)x + b(R_n).$$

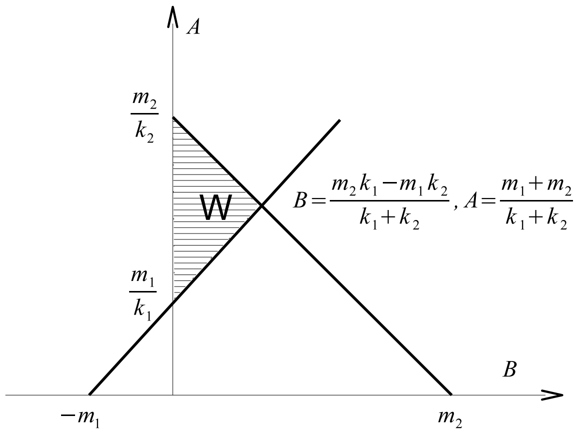
We use these bounds to estimate the stopping time that appears in Lemma 3.3. In particular as an immediate consequence of the lemma above we obtain

Corollary 3.4. *Let*

$$W = W(m_1, m_2, k_1, k_2) = \{(B, A) \in \mathrm{Aff}(\mathbb{R}) \mid Ak_2 + B \leq m_2; \quad Ak_1 - B \geq m_1\}$$

and $T_W = \inf\{n \geq 0 : R_n \in W\}$. *Then we have*

$$\nu(m_1, m_2) \geq \mathbb{P}[T_W < \infty]\nu(k_1, k_2).$$



Proof. The Corollary follows from Lemma 3.3, taking $U = [k_1, k_2]$, $V = [m_1, m_2]$ and noticing that $T_W \geq T_{\mathfrak{W}}$. □

Since the potential theory of the affine group is well understood, we have enough tools to estimate $\mathbb{P}(T_W < +\infty)$ in many situations. For a continuous and compactly supported function f on $\mathrm{Aff}(\mathbb{R})$ we define the potential

$$U * \delta_g(f) := \mathbb{E}\bigg[\sum_{n=0}^{\infty} f(L_n g)\bigg] = \mathbb{E}\bigg[\sum_{n=0}^{\infty} f(R_n g)\bigg].$$

A renewal theorem for the potential U, i.e. description of its behavior at infinity, was given in [2], where the authors proved that for all $h \in C_C(\mathrm{Aff}(\mathbb{R}))$:

$$\lim_{a\to 0} U * \delta_{(0,a)}(h) = \nu_G \otimes \frac{dx}{x}(h), \tag{3.5}$$

for ν_G being a suitable non-trivial multiple of the invariant measure of the process $Y_n^x = L_n(x)$.

Now we are ready to prove the following lemma.

Lemma 3.6. *Suppose (1.4), (1.5), (2.3) and (2.4). There exist a compact subset*

$$V_0 = \{(B, A) \in \mathrm{Aff}(\mathbb{R}) \mid |B| < b_0, a_0^{-1} < A < a_0\}$$

and a constant $\delta > 0$ such that:

(1) if $W_z = (0, z)\cdot V_0 = \{(B, A) \mid |B| < z\, b_0,\ z\, a_0^{-1} < A < z\, a_0\}$, then

$$\mathbb{P}(T_{W_z} < \infty) > \delta$$

for all $z \ge 1$;

(2) if $V_z = V_0 \cdot (0, z^{-1}) = \{(B, A) \mid |B| < b_0,\ a_0^{-1}/z < A < a_0/z\}$, then

$$\mathbb{P}(T_{V_z} < \infty) > \frac{\delta}{1 + \log z}$$

for all $z \ge 1$.

Proof. Step 1. First observe that for every $V \subset \mathrm{Aff}(\mathbb{R})$

$$U(V^{-1}V)\mathbb{P}(T_V < \infty) \ge U(V). \tag{3.7}$$

In fact

$$U(V) = \sum_{n=0}^{\infty} \mathbb{P}[R_n \in V] = \mathbb{E}\Big[\mathbf{1}_{\{T_V<\infty\}} \sum_{n=T_V}^{\infty} \mathbf{1}_{\{R_{T_V} R_n^{T_V} \in V\}}\Big] \le \mathbb{P}(T_V < \infty)U(V^{-1}V)$$

where $R_n^l := R_l^{-1} R_n = g_{l+1} \cdots g_n$.

Step 2: Proof of (1). By (3.7) we write (assuming the denominator is nonzero)

$$\mathbb{P}(T_{W_z} < \infty) \ge \frac{U(W_z)}{U(W_z^{-1}W_z)} = \frac{U((0, z)\cdot V_0)}{U(V_0^{-1}V_0)}. \tag{3.8}$$

A simple calculation relates the right random walk on the affine group to the reversed left random walk $\check{L}_n = R_n^{-1} = g_n^{-1} \cdots g_1^{-1}$. Observe that for any $V \subset \mathrm{Aff}(\mathbb{R})$ we have

$$\begin{aligned} U((0, z)V) &= \sum_n \mathbb{P}\big[R_n \in (0, z)V\big] = \sum_n \mathbb{P}\big[R_n^{-1} \in V^{-1}(0, z^{-1})\big] \\ &= \sum_n \mathbb{P}\big[\check{L}_n(0, z) \in V^{-1}\big] = \check{U}\big(V^{-1}(0, z^{-1})\big), \end{aligned}$$

where $\check{U}$ is the potential of the reversed random walk $\check{L}_n$. Since the law of g_n^{-1} is also centered and verifies the hypotheses of [2], there exists a unique Radon measure $\check{\nu}_G$ on $\mathbb{R}$ invariant under $\check{\mu}_G$, the law of $g^{-1} = (B, A)^{-1}$. Then by (3.5)

$$\lim_{z\to+\infty} U((0, z)V) = \lim_{z\to+\infty} \check{U}\big(V^{-1}(0, z^{-1})\big) = \Big(\check{\nu}_G \times \frac{dx}{x}\Big)(V^{-1}).$$

We take sufficiently large V_0 such that

$$U(W_z^{-1}W_z) = U(V_0^{-1}V_0) > 0 \quad \text{and} \quad \Big(\check{\nu}_G \times \frac{dx}{x}(V_0^{-1})\Big) > 0$$

and, in view of (3.7), we conclude.

Step 3: Proof of (2). As in the previous step, by (3.7), we write

$$\mathbb{P}(T_{V_z} < \infty) > \frac{U(V_z)}{U(V_z^{-1}V_z)} = \frac{U\big(V_0(0,z^{-1})\big)}{U\big((0,z)V_0^{-1}V_0(0,z^{-1})\big)} \tag{3.9}$$

Now we have to estimate $U(V_z)$ from below and $U(V_z^{-1}V_z)$ from above. The later is the most difficult part of the proof.

To deal with this second problem we decompose the centered random walk on the affine group in a contracting part and a dilating part using ladder stopping times. This key idea has been applied in several different ways in important works on the subject, for instance [16, 13, 19, 2]. We use here a potential version. Let $\{\overline{g}_i\}$ be another sequence of i.i.d. elements of Aff($\mathbb{R}$) independent and of the same law as $\{g_i\}$. We define $\overline{S}_n, \bar{t}_k, \bar{l}_k$ as in (2.6) and (2.7). We claim that

$$U(f) = \mathbb{E}\Big[\sum_{n=0}^{\infty} f(L_n)\Big] = \mathbb{E}\Big[\sum_{k,i=0}^{\infty} f(\overline{R}_{\bar{l}_i} L_{t_k})\Big]. \tag{3.10}$$

In fact, for $n > k$ define $L_n^k = g_n \cdots g_{k+1}$ and $L_k^k = e$. Observe that

$$\mathbb{E}\Big[\sum_{n=0}^{\infty} f(L_n)\Big] = \mathbb{E}\Big[\sum_{k=0}^{\infty}\sum_{i=t_k}^{t_{k+1}-1} f(L_i)\Big] = \mathbb{E}\Big[\sum_{k=0}^{\infty}\mathbb{E}\Big[\sum_{i=t_k}^{t_{k+1}-1} f(L_i^{t_k} L_{t_k})\Big| L_{t_k}\Big]\Big].$$

Since for fixed k the sequence $\{L_{t_k+i}^{t_k}\}_{i\geq 0}$ is independent of L_{t_k} and has the same law as $\{L_i\}_{i\geq 0}$, by the duality lemma (see lemma 5.4 [6]) we have

$$\mathbb{E}\Big[\sum_{i=t_k}^{t_{k+1}-1} f(L_i^{t_k} L_{t_k}) | L_{t_k} = g\Big] = \mathbb{E}\Big[\sum_{i=0}^{t_1-1} f(L_i g)\Big] = \mathbb{E}\Big[\sum_{i=0}^{\infty} f(\overline{R}_{\bar{l}_i} g)\Big]$$

and we obtain (3.10).

Observe that $\overline{S}_{\bar{l}_i}$ (resp. S_{t_k}) is a random walk with finite mean and negative (resp. positive) steps. Take $a, b > 2$, then by (3.10) and the classical renewal theorem [14], we have

$$\begin{aligned}
U([-b,b]\times[1/a,a]) &= \sum_{k,i=0}^{\infty} \mathbb{P}\big[b(\overline{R}_{\bar{l}_i} L_{t_k}) \leq b; -\log a \leq \overline{S}_{\bar{l}_i} + S_{t_k} \leq \log a\big]\\
&= \sum_{k,i=0}^{\infty} \mathbb{P}\big[\mathrm{e}^{-\overline{S}_{\bar{l}_i}} b(L_{t_k}) + b(\overline{R}_{\bar{l}_i}) \leq b; -\log a \leq \overline{S}_{\bar{l}_i} + S_{t_k} \leq \log a\big]\\
&\leq \sum_{k,i=0}^{\infty} \mathbb{P}\big[b(\overline{R}_{\bar{l}_i}) \leq b; -\log a \leq \overline{S}_{\bar{l}_i} + S_{t_k} \leq \log a\big] \quad \text{since } b(L_{t_k}) \geq 0\\
&= \sum_{i=0}^{\infty} \mathbb{E}\Big[\mathbf{1}_{[b(\overline{R}_{\bar{l}_i})\leq b]} \mathbb{E}\Big[\sum_{k=0}^{\infty} \mathbf{1}_{\{-\log a \leq \overline{S}_{\bar{l}_i} + S_{t_k} \leq \log a\}}\Big| \overline{g_i}, i \geq 0\Big]\Big]\\
&\leq C \log a \sum_{i=0}^{\infty} \mathbb{P}\big[b(\overline{R}_{\bar{l}_i}) \leq b\big]
\end{aligned}$$

Since we assume $B \geq 1$ a.s., we have for $i \geq 1$:

$$b(\overline{R}_{\bar{l}_i}) = b(\overline{R}_{\bar{l}_{i-1}} \overline{R}_{\bar{l}_i}^{\bar{l}_{i-1}}) = \mathrm{e}^{-\overline{S}_{\bar{l}_{i-1}}} b(\overline{R}_{\bar{l}_i}^{\bar{l}_{i-1}}) + b(\overline{R}_{\bar{l}_{i-1}}) \geq \mathrm{e}^{-\overline{S}_{\bar{l}_{i-1}}}$$

That is

$$U([-b,b]\times[1/a,a]) \le C\log a\bigg(1+\sum_{i=1}^{\infty}\mathbb{P}\big[\overline{S}_{\overline{l}_{i-1}} \ge -\log b\big]\bigg) \le C\log a\,(1+C\log b)$$

Therefore, since

$$V_z^{-1}V_z \subseteq \{(B,A) \mid |B| \le 2b_0a_0\,z,\ a_0^{-2} \le A \le a_0^2\},$$

we obtain

$$U(V_z^{-1}V_z) \le K\log a_0(1+\log z+\log(2b_0a_0)).$$

To estimate $U(V_z)$ from below as in the previous case, we just apply the renewal theorem (3.5). Plugging those estimates into (3.9), we conclude. □

Proof of Proposition 3.1.

Step 1: Proof of (1). We apply Corollary 3.4 with $[k_1,k_2]=[-z,z]$ and $[m_1,m_2]=[-2b_0,2b_0]$ and consider, according to the notation there, the subset of $\mathrm{Aff}(\mathbb{R})$

$$W(-2b_0,2b_0,-z,z)=\big\{g\in \mathrm{Aff}(\mathbb{R}) \mid g([-z,z])\subseteq[-2b_0,2b_0]\big\}=\big\{(B,A) \mid\ Az+B<2b_0\big\}.$$

This subset contains the set

$$V_z=\bigg\{(B,A)\Big|\frac{b_0^{-1}}{z}<A<\frac{b_0}{z},\ |B|<b_0\bigg\}.$$

We can apply Corollary 3.4 and, choosing b_0 large enough, Lemma 3.6.2 to conclude:

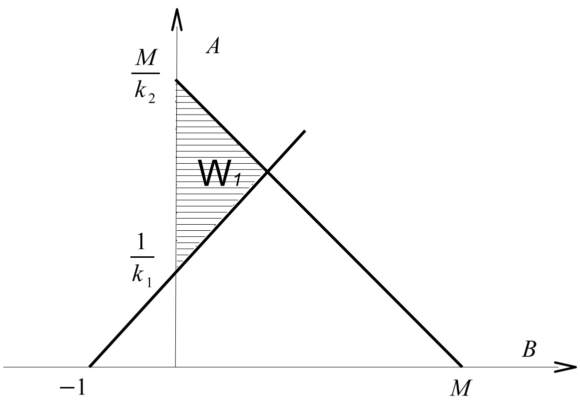
$$\nu(-z,z)\le \frac{\nu[-2b_0,2b_0]}{\mathbb{P}(T_{V_z}<\infty)} < C_0(1+\log z).$$

Step 2: Proof of (2). Take $M>1$ and $0<k_1<k_2$. Set $[m_1,m_2]=[z,zM]$. Then by Corollary 3.4

$$\nu[z,zM]\ge \mathbb{P}(T_{W_z}<\infty)\nu[k_1,k_2],$$

where

$$W_z=W(z,zM,k_1,k_2)=(0,z)W(1,M,k_1,k_2)=:(0,z)W_1.$$



Observe that if k_1, M and M/k_2 tend to infinity, then

$$W_1=\{(B,A) \mid\ Ak_1-B>1, Ak_2+B<M\}$$

grows to $\mathrm{Aff}(\mathbb{R})$. Thus, there exists $C>0$ such that if $k_1\ge C$,$M>C$ and $M/k_2\ge C$, the set W_1 contains the compact set V_0 defined in Lemma 3.6. Therefore $\mathbb{P}(T_{W_z}<\infty)$ is uniformly bounded from below for large values of z. Moreover, since we require the support of ν to be unbounded on the right, one can choose k_2 such that $\nu[k_1, k_2] > 0$ and we conclude.

STEP 3: PROOF OF (3). Let a_0, b_0 be sufficiently large numbers such that Lemma 3.6 holds. Take $M > \max\{2, 4a_0^2\}$.

First suppose that $\frac{u_2}{u_1} < \frac{M}{4a_0^2}$. Take $[m_1, m_2] = [\mathrm{e}^x, \mathrm{e}^x M]$ and $[k_1, k_2] = [\mathrm{e}^{x+y}u_1, \mathrm{e}^{x+y}u_2]$. For $x > \log(b_0)$, the set

$$W(\mathrm{e}^x, \mathrm{e}^x M, \mathrm{e}^{x+y}u_1, \mathrm{e}^{x+y}u_2) = \Big\{(B, A) \in \mathrm{Aff}(\mathbb{R}) \mid A\mathrm{e}^{x+y}u_2 + B \le \mathrm{e}^x M; \quad A\mathrm{e}^{x+y}u_1 - B \ge \mathrm{e}^x\Big\}$$

contains the set

$$V(y) = \Big\{(B, A) \in \mathrm{Aff}(\mathbb{R}) \mid B < b_0, \frac{2}{\mathrm{e}^y u_1} \le A \le \frac{M}{\mathrm{e}^y 2u_2}\Big\}.$$

Since $(\frac{M}{\mathrm{e}^y 2u_2})/(\frac{2}{\mathrm{e}^y u_1}) = \frac{Mu_1}{4u_2} > a_0^2$, we can apply Lemma 3.6 and prove that there exists $C >$ such that

$$\frac{\nu[\mathrm{e}^{x+y}u_1, \mathrm{e}^{x+y}u_2]}{\nu[\mathrm{e}^x, \mathrm{e}^x M]} \le \frac{1}{\mathbb{P}(T_{V(y)} < \infty)} < C(1+y) \quad \text{for all } x > \log b_0, y > 0$$

By the previous steps, the last inequality is satisfied for $0 < x \le \log b_0$ and all $y > 0$.

For general $U = [u_1, u_2]$ with $\frac{u_2}{u_1} \ge \frac{M}{4a_0^2}$ we can deduce (3.2) covering U with a finite number of small intervals.

□

Since the law of $\log A$ is aperiodic, proceeding as in [2] and [6], one can prove that the family of quotient measures is asymptotically invariant under the action of $\mathbb{R}_+^*$ and converges to the Haar measure of $\mathbb{R}_+^*$.

Corollary 3.11. *Under the hypotheses of Theorem 1.3*

$$\liminf_{z\to\infty} \delta_{z^{-1}} * \nu(\phi) > 0,$$

where ϕ is an arbitrary nonzero and nonnegative element of $C_c(0, +\infty)$.

Furthermore for $\phi_1, \phi_2 \in C_c(0, +\infty)$ and ϕ_2 not identically zero:

$$\lim_{z\to\infty} \frac{\delta_{z^{-1}} * \nu(\phi_1)}{\delta_{z^{-1}} * \nu(\phi_2)} = \frac{\int_{\mathbb{R}_+^*} \phi_1(a)\frac{da}{a}}{\int_{\mathbb{R}_+^*} \phi_2(a)\frac{da}{a}}. \tag{3.12}$$

Therefore

$$\lim_{x\to+\infty} \frac{\delta_{e^{-(x+y)}} * \nu(\phi)}{\delta_{e^{-x}} * \nu(\phi)} = 1 \tag{3.13}$$

and

$$\frac{\delta_{e^{-(x+y)}} * \nu(\phi)}{\delta_{e^{-x}} * \nu(\phi)} \le K_\phi(1+y) \text{ for all } x, y > 0.$$

In particular the function $L(z) = \delta_{z^{-1}} * \nu(\phi)$ is slowly varying.

Proof. For the reader's convenience we present a sketchy proof (see Proposition 2.2 [6] for more details). First take a Lipschitz function Φ whose compact support contains $(1, M)$ and let $L(z) = \delta_{z^{-1}} * \nu(\Phi)$. Since the family of measures $\widetilde{\nu}_z = \frac{1}{L(z)}\delta_{z^{-1}} * \nu$ is vaguely compact, for every sequence we can extract its subsequence $\widetilde{\nu}_{z_n}$ convergent to a limit measure η.

For every Lipschitz compactly supported function ϕ and $\Psi \in \mathfrak{F}$ there exists a compact set $U = U(\phi, \Psi)$ such that

$$\left|\phi\Big(\frac{\Psi(u)}{z}\Big) - \phi\Big(\frac{Au}{z}\Big)\right| \le \frac{B}{z} \cdot \mathbf{1}_U\Big(\frac{Au}{z}\Big),$$

and

$$\begin{aligned}\lim_{n\to\infty}\frac{\left|\int\phi\Big(\frac{\Psi(u)}{z_n}\Big)\nu(du)-\int\phi\Big(\frac{Au}{z_n}\Big)\nu(du)\right|}{L(z_n)} &\le \lim_{n\to\infty}\frac{C|z_n^{-1}b|\nu(a^{-1}z_nU)}{L(z_n)}\\ &\le C\eta(a^{-1}U)\cdot\lim_{n\to\infty}|z_n^{-1}b|=0,\end{aligned}$$

Thus the function

$$h(y)=\delta_y*\eta(\phi)=\lim_{n\to\infty}\frac{\delta_{(0,z_n^{-1}y)}*_G\nu(\phi)}{L(z_n)}$$

on $\mathbb{R}_+^*$ is superharmonic with respect to the action of μ_A, the law A_1. Since h is positive and continuous, by the Choquet-Deny theorem it must be a constant function, that is $\delta_a*\eta(\phi)=\eta(\phi)$ for every $a\in\mathbb{R}_+^*$. Because $\eta(\Phi)=1$, then η is a fixed multiple of the Haar measure of $\mathbb{R}_+^*$ and

$$\lim_{z\to+\infty}\frac{\delta_{z^{-1}}*\nu(\phi)}{\delta_{z^{-1}}*\nu(\Phi)}=\frac{\int\phi(a)\frac{da}{a}}{\int\Phi(a)\frac{da}{a}}.$$

This proves (3.12) and (3.13). In particular, if ϕ is nonzero, by Proposition 3.1, we have

$$\liminf_{z\to\infty}\delta_{z^{-1}}*\nu(\phi)\ge\int\phi(a)\frac{da}{a}\cdot\liminf_{z\to\infty}\delta_{z^{-1}}*\nu(\Phi)>0.$$

Take k such that the support of ϕ is contained in $[1/k,k]$. Then

$$\frac{e^{-(x+y)}*\nu(\phi)}{e^{-x}*\nu(\phi)}\le\frac{\nu[\mathrm{e}^x/M,\mathrm{e}^xM]}{e^{-x}*\nu(\phi)}\frac{\nu[\mathrm{e}^{x+y}/k,\mathrm{e}^{x+y}k]}{\nu[\mathrm{e}^x/M,\mathrm{e}^xM]}\le K(1+y),$$

because the first quotient is bounded. □

4. Homogeneity at infinity

In this section we finish the proof of Theorem 1.3. The main idea of the proof is similar to our previous papers [6, 8, 7]. Given a nice function ϕ on $\mathbb{R}_+^*$ we define the function

$$f(x)=\int_{\mathbb{R}_+^*}\phi(e^{-x}u)\nu(du).$$

Behavior at infinity of the measure ν is coded in the asymptotic behavior of f. To describe f we consider it as a solution of the Poisson equation

$$\overline{\mu}*_{\mathbb{R}}f(x)=f(x)+g(x)$$

where $\overline{\mu}$ is the law of $-\log A$ and the function g is defined by the equation above. We cannot use the classical renewal theorem, since the measure $\overline{\mu}$ is centered. In our previous papers we expressed f as a special potential of g. However this approach was technically involved and it was not possible to establish the optimal hypotheses. Here we apply ideas due to Durrett and Liggett [12], who studying similar equation and applying the duality lemma, were able to reduce the problem to the classical renewal theorem. In Proposition 4.1, we determine weak assumptions in the terms of the Poisson equation that enable to control the asymptotic behavior of the solution.

In the second part of the section, we apply this result to our problem. We show that there exist slight perturbations of the functions f and g defined above which satisfy all the required conditions. Finally we deduce our main result proving that the tail of the measure ν converges at infinity.

Proposition 4.1. *Let $\overline{\mu}$ be a centered probability measure on $\mathbb{R}$ with finite moment of order $2+\epsilon$ for some $\epsilon > 0$ and let f be a continuous function on $\mathbb{R}$ such that*

$$0 \le f(x) \le C(1+x^+) \qquad \text{and} \qquad \int_{-\infty}^{y} f(x)dx \le C(1+y^+) \tag{4.2}$$

where $x^+ := \max\{0,x\}$. Let g be the continuous function on $\mathbb{R}$ defined by the Poisson equation:

$$\overline{\mu} * f(x) = f(x) + g(x). \tag{4.3}$$

Suppose also that g is directly Riemann integrable, then

$$\lim_{x\to+\infty} \mathbb{E}[f(x+S_t)] - f(x) = \frac{-1}{\mathbb{E}[S_l]} \int_{\mathbb{R}} g(x)dx, \tag{4.4}$$

where S_n is the random walk of law $\overline{\mu}$ and t and l are the stopping times

$$t = \inf\{n > 0 : S_n \ge 0\} \text{ and } l = \inf\{n > 0 : S_n < 0\}.$$

Moreover, if $\int_{\mathbb{R}} g(x)dx = 0$ and $\int_{\mathbb{R}} |xg(x)|dx < \infty$,

$$\lim_{x\to+\infty} \mathbb{E}\left[\int_x^{x+S_t} f(z)dz\right] = \frac{1}{\mathbb{E}[S_l]} \int_{\mathbb{R}} x\, g(x)dx. \tag{4.5}$$

The notion of directly Riemann integrable functions is fundamental in the renewal theory and allows to apply the classical renewal theorem to the function g (see for e.g. Feller [14]). The proof of this proposition will be given in Appendix A.

Let ν be a μ-invariant Radon measure on $\mathbb{R}$. We would like to apply the previous proposition to the function $f(x) = \delta_{e^{-x}} * \nu(\phi)$ for some fixed positive function $\phi \in C^1_C(\mathbb{R}^*_+)$. Unfortunately we are not able to justify that f satisfies all the required hypotheses. The main reason is that we are not able to control local properties of a general measure ν, namely its behavior near 0. Thus the function f may not be sufficiently integrable at $-\infty$. However it turns out that one can slightly translate the measure ν to overcome the problem.

For this purpose, given $\phi \in C^1_C(\mathbb{R}^*_+)$ and $w_0 > 0$ define

$$\begin{aligned} f_\phi(x) &:= \int_{\mathbb{R}} \phi(e^{-x}(u-w_0))\nu(du), \\ g_\phi(x) &:= \overline{\mu} *_{\mathbb{R}} f_\phi(x) - f_\phi(x). \end{aligned}$$

Observe that $f_\phi(x) = \delta_{e^{-x}} * \nu_0(\phi)$ where ν_0 is the measure ν translated by w_0:

$$\nu_0(\phi) = \int_{\mathbb{R}} \phi(u-w_0)\nu(du),$$

i.e. the invariant measure of the SDS obtained by conjugating the original one with the translation by w_0:

$$\psi_0(x) = \psi(x+w_0) - w_0.$$

Denote by μ_0 its law. Observe that $A(\psi_0) = A(\psi)$ and we can choose $B(\psi_0) = Aw_0 + w_0 + B$, hence μ_0 satisfies our main hypotheses if μ does. Since the translation doesn't change the asymptotic behavior, the measures ν_0 and ν behave in the same way at $+\infty$, namely :

$$\lim_{x\to+\infty} f_\phi(x) - \delta_{e^{-x}} * \nu(\phi) = 0. \tag{4.6}$$

In fact

$$\begin{aligned} \int_{-\infty}^{+\infty} |\phi(e^{-x}(u-w_0)) - \phi(e^{-x}u)|\nu(du) &\le C\int_0^{\infty} |e^{-x}w_0|\mathbf{1}_{[e^x m, e^x(M+w_0)]}\nu(du) \\ &\le C|e^{-x}w_0|\log(e^x(M+w_0)) \end{aligned}$$

when $\text{supp}(\phi) \subset [m, M]$. Summarizing, translation of the invariant measure does not change the problem we study, nor our assumptions. Existence of a corresponding w_0 is provided by the following lemma, whose proof will be given in Appendix A.

Lemma 4.7. *There exists $w_0 > 0$ such that for all $\phi \in C^1_C(\mathbb{R}^*_+)$ the functions f_ϕ and g_ϕ satisfy the hypotheses of Proposition 4.1.*

Now we are ready to prove our main result.

Proof of Theorem 1.3. We claim that $\int g_\phi(y)dy = 0$. In fact for all y we can apply Corollary 3.11

$$\lim_{x\to+\infty} \frac{f_\phi(x+y)}{f_\phi(x)} = \lim_{x\to+\infty} \frac{\delta_{\mathrm{e}^{-(x+y)}} * \nu_0(\phi)}{\mathrm{e}^{-x} * \nu_0(\phi)} = 1;$$

thus, since $\mathbb{E}(S_t)$ is finite, by dominated convergence $\mathbb{E}(f_\phi(x+S_t)/f_\phi(x))$ also converges to 1. Fix $\varepsilon > 0$, then there exists x_ε such that for all $x \geq x_\varepsilon$

$$\left|\mathbb{E}\big[f_\phi(x+S_t)\big] - f_\phi(x) + \frac{1}{\mathbb{E}S_t}\int g_\phi(y)dy\right| < \varepsilon \text{ and } \left|\frac{\mathbb{E}[f_\phi(x+S_t)]}{f_\phi(x)} - 1\right| < \varepsilon.$$

Therefore $f_\phi(x) \geq |\int g_\phi(y)dy|/(\varepsilon\mathbb{E}S_t) - 1$. Since by Lemma 4.7, $\int_{-\infty}^x f_\phi(y)dy < C(1+x)$, for all $x > x_\varepsilon > 0$

$$C(1+x) \geq \int_{x_\varepsilon}^x f_\phi(y)dy \geq \left(\frac{|\int g_\phi(y)dy|}{\varepsilon\mathbb{E}S_t} - 1\right)(x - x_\varepsilon).$$

That is

$$\left|\int_{\mathbb{R}} g_\phi(y)dy\right| \leq \varepsilon\mathbb{E}S_t\left(\liminf_{x\to+\infty} \frac{C(1+x)}{x-x_\varepsilon} + 1\right) = \varepsilon\mathbb{E}S_t(C+1).$$

Letting $\varepsilon \searrow 0$, we conclude.

In view of Corollary 3.11, the quotient $f_\phi(x+y)/f_\phi(x)$ is uniformly dominated by $1 + S_t$ for $x > 0$ and $0 < y < S_t$, thus

$$\lim_{x\to\infty} \int_0^{S_t} \frac{f_\phi(x+y)}{f_\phi(x)} dy = \int_0^{S_t} 1\ dy = S_t. \qquad \mathbb{P} \text{ a.s.}$$

By Fatou's lemma

$$\liminf_{x\to\infty} \mathbb{E}\left[\int_0^{S_t} \frac{f_\phi(x+y)}{f_\phi(x)} dy\right] \geq \mathbb{E}\left[\liminf_{x\to\infty} \int_0^{S_t} \frac{f_\phi(x+y)}{f_\phi(x)} dy\right] = \mathbb{E}[S_t] \tag{4.8}$$

Therefore by Proposition 4.1

$$\limsup_{x\to\infty} f_\phi(x) = \limsup_{x\to\infty} \frac{\mathbb{E}\left[\int_0^{S_t} f_\phi(x+y)dy\right]}{\mathbb{E}\left[\int_0^{S_t} \frac{f_\phi(x+y)}{f_\phi(x)} dy\right]} \leq \frac{1}{\mathbb{E}[S_l]\mathbb{E}[S_t]} \int_{\mathbb{R}} g_\phi(x)xdx$$

In particular this proves that $f_\phi(x)$ is bounded above. Since by Corollary 3.11, we already know that $f_\phi(x)$ is bounded below, $\int_0^{S_t} \frac{f_{\phi_i}(x+y)}{f_\phi(x)} dy < CS_t$. This allow to use the dominated convergence theorem instead of Fatou's Lemma in (4.8) and to replace the inferior limit with the real limit and the inequality with the equality. Thus we have

$$\lim_{x\to\infty} f_\phi(x) = \lim_{x\to\infty} \frac{\mathbb{E}\left[\int_0^{S_t} f_\phi(x+y)dy\right]}{\mathbb{E}\left[\int_0^{S_t} \frac{f_\phi(x+y)}{f_\phi(x)} dy\right]} = \frac{1}{\mathbb{E}[S_l]\mathbb{E}[S_t]} \int_{\mathbb{R}} g_\phi(x)xdx = \frac{1}{\sigma^2} \int_{\mathbb{R}} g_\phi(x)xdx,$$

where $\sigma^2 = \int x^2 \overline{\mu}(dx)$ (see [14] for the proof that $\mathbb{E}[S_l]\mathbb{E}[S_t] = \sigma^2$).

To conclude take a nonzero nonnegative function $\Phi \in C_c^1(0,+\infty)$. We have proved that the following limit exists

$$\lim_{z\to+\infty} \delta_{z^{-1}} * \nu(\Phi) = \lim_{x\to+\infty} f_\Phi(x) = C$$

and by Corollary 3.11 the constant C is strictly positive. The same corollary also implies that for all $\phi \in C_c(0,+\infty)$

$$\lim_{z\to+\infty} \delta_{z^{-1}} * \nu(\phi) = \lim_{z\to+\infty} \frac{\delta_{z^{-1}} * \nu(\phi)}{\delta_{z^{-1}} * \nu(\Phi)} \lim_{z\to+\infty} \delta_{z^{-1}} * \nu(\Phi) = \frac{C}{\int_{\mathbb{R}} \Phi(a)\frac{da}{a}} \int_{\mathbb{R}} \phi(a)\frac{da}{a}.$$

□

5. Uniqueness of the invariant measure

Proof of Theorem 1.9. Notice first that for any compact set K

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{1}_K(X_n^y)|X_n^y - X_n^{y'}| &\leq |y-y'| \limsup_{n\to\infty} A_1 \dots A_n \mathbf{1}_K(X_n^y) \\
&= |y-y'| \limsup_{n\to\infty} \frac{X_n^y \mathbf{1}_K(X_n^y)}{\frac{X_n^y}{A_1\cdots A_n}} \\
&\leq \limsup_{n\to\infty} \frac{C(K)}{\frac{X_n^y}{A_1\cdots A_n}}.
\end{aligned}$$

Thus it is sufficient to prove that

$$\lim_{n\to\infty} \frac{X_n^y}{A_1\cdots A_n} = +\infty.$$

Notice that the sequence $\frac{X_n^y}{A_1\cdots A_n}$ in nondecreasing. Indeed, since $\Psi_n(X_{n-1}^y) \geq A_n X_{n-1}^y$,

$$\frac{X_n^y}{A_1\cdots A_n} = \frac{\Psi_n(X_{n-1}^y)}{A_1\cdots A_n} \geq \frac{X_{n-1}^y}{A_1\cdots A_{n-1}}.$$

Therefore it is enough to justify that for arbitrary large fixed $M>0$ the sequence is a.s. at least once greater than M. Let

$$U_{\beta,\gamma} := \{\Psi \in \mathfrak{F} | \Psi[0,+\infty) \subseteq [\beta,+\infty) \text{ and } A(\Psi) < \gamma\}$$

and

$$V_\alpha := \{\Psi \in \mathfrak{F} | A(\Psi) < \alpha\}.$$

In view of our hypotheses there exist $\alpha < 1$, $\beta > 0$, and γ such that these two sets have positive probability. For a fixed x_0, take $N>0$ such that $\alpha^{N-1} M\gamma x_0 < \beta$ and let $\psi_0 = \psi_1\psi_2$ with $\psi_1 \in U_{\beta,\gamma}$ and $\psi_2 \in V_\alpha^{N-1}$. We claim that

$$\frac{\psi_0(x)}{A(\psi_0)x} > M \text{ for all } 0 \leq x \leq x_0. \tag{5.1}$$

In fact

$$\psi_0(x) = \psi_1(\psi_2(x)) \geq \beta > M(\gamma\alpha^{N-1}x_0) > MA(\psi_1)A(\psi_2)x > MA(\psi_0)x.$$

Observe that since X_n^y is recurrent, there exists $x_0 > 1$ such that $\mathbb{P}[0 \leq X_n^y < x_0 \ i.o.] = 1$ for every $y \geq 0$. Let us fix y, x_0 and define a sequence T_k of hitting times of $[0, x_0]$

$$\begin{aligned}
T_0 &= 0, \\
T_k &= \inf\left\{n > T_{k-1} + N : X_n^y < x_0\right\}.
\end{aligned}$$

By recurrence, all T_k are almost surely finite. Let $\Psi_i^j := \Psi_j \circ \cdots \circ \Psi_{i+1}$, then $\{\Psi_{T_k}^{T_k+N}\}$ is a sequence of i.i.d. random transformations distributed as μ^N. Since $\mu^N(U_{\beta,\gamma}V_\alpha^{N-1}) > 0$ there exists almost surely k_0 such that $\Psi_{T_{k_0}}^{T_{k_0}+N} \in U_{\beta,\gamma}V_\alpha^{N-1}$. Then, by (5.1), we have:

$$\frac{X^y_{T_{k_0}+N}}{A_1 \cdots A_{T_{k_0}+N}} = \frac{\Psi_{T_{k_0}}^{T_{k_0}+N}(X^y_{T_{k_0}})}{A_1 \cdots A_{T_{k_0}+N}} \geq \frac{\Psi_{T_{k_0}}^{T_{k_0}+N}(X^y_{T_{k_0}})x_0}{A_{T_{k_0}+1} \cdots A_{T_{k_0}+N} X^y_{T_{k_0}}} > M.$$

□

6. Examples

In this section we present some of the more significant classes of stochastic dynamical system to which the results of the previous sections apply.

6.1. **The random difference equation.** The first example is naturally the SDS induced by random affinities, that is $\Psi_n(x) = A_n x + B_n$, for a random pair $(B_n, A_n) \in \mathbb{R} \times \mathbb{R}_+^*$. Then X_n^x is given by formula (1.7). This process is called the random difference equation or the affine recursion. It is well known that under the assumptions of Theorem 1.3 this process is recurrent and locally contractive, thus it possesses a unique invariant Radon measure ν, see [2]. Behavior of this measure at infinity was studied previously in [8, 6, 7] under a number of additional strong hypotheses. Theorem 1.3 provides an optimal result, in the sense that the hypotheses implying existence and uniqueness of the invariant measure, are sufficient also to deduce that this measure must behave at infinity like $\frac{Cdx}{x}$.

6.2. **Stochastic recursions with unique invariant measure.** Our results can also be applied to a more general class of stochastic recursions that behave at infinity as Ax (i.e. $\Phi(x) \sim Ax$ for large x). In the contracting case ($\mathbb{E}[\log A] < 0$) those recursions were studied by Goldie [15] (see also Mirek [21], who described this class of recursions in general settings, including more examples). Just to give some concrete examples let us mention that our results are valid (under rather obvious and easy to formulate assumptions) for the following examples

- $\Psi_{1,n}(x) = \max\{A_n x, B_n\} + C_n$, for $A_n, B_n, C_n > 0$.
- $\Psi_{2,n}(x) = \sqrt{A_n^2 x^2 + B_n x + C_n}$, for $A_n, B_n, C_n > 0$ and $\Delta = B^2 - 4A^2C \leq 0$

In both cases above the mappings $\Psi_{i,n}$ are Lipschitz with the Lipschitz coefficient equal to A. This is obvious for the first example. For the second one, denote $x_0 = -\frac{B}{2A^2}$, $D = -\frac{\Delta}{4A^2}$. Observe that since $\Psi_{2,n}(x) = \sqrt{A^2(x-x_0)^2 + D}$, its derivative

$$\Psi'_{2,n}(x) = \frac{A^2(x-x_0)}{\sqrt{A^2(x-x_0)^2 + D}} = A\frac{1}{\sqrt{1 + \frac{D}{A^2(x-x_0)^2}}} \nearrow A$$

is an increasing function that tends to A. Hence, under appropriate moment assumptions, the SDS on $\mathbb{R}_+$ generated by the random functions defined above satisfies assumptions of both Theorem 1.3 and Theorem 1.9. Therefore the corresponding random process possesses a unique invariant measure, which behaves at infinity like $\frac{Cdx}{x}$.

If we do not suppose $\Delta = B^2 - 4A^2C \leq 0$, then $\Psi_{2,n}$ are still asymptotically linear functions to which Theorem 1.3 applies, but we cannot prove uniqueness of an invariant measure.

6.3. **Random automorphisms of the interval** $[0,1]$**.** SDS's acting on the real line after conjugating by an appropriate function can be seen as random automorphisms of the interval $[0,1]$ fixing the end points. Our key property (AL) is translated in this setting into requiring that the automorphisms "reflect" at the same way in 0 and in 1, in the sense that the derivative in these two points has to be the same. The B term is then related to the term of order two at these end points (or order $2-\alpha$, if we conjugate a SDS that satisfy (AL^α)). More precisely

Corollary 6.1. *Consider a SDS on* $[0,1]$ *defined by random functions* $\phi \in C([0,1])$ *fixing* 0 *and* 1*, differentiable at the extremities of the interval and such that*

$$\phi'(0) = \phi'(1) =: a_\phi.$$

Let

$$\beta_1^0 = \inf_{u\in[0,1/2]}(1-\phi(u)) > 0, \quad \beta_2^0 = \inf_{u\in[0,1/2]} \frac{\phi(u)}{u} > 0, \quad \beta_3^0 = \sup_{u\in[0,1/2]} \left|\frac{\phi(u)-a_\phi u}{u^2}\right| < \infty.$$

$$\beta_1^1 = \inf_{u\in[1/2,1]} \phi(u) > 0, \ \beta_2^1 = \inf_{u\in[1/2,1]} \frac{1-\phi(u)}{1-u} > 0, \ \beta_3^1 = \sup_{u\in[1/2,1]} \left|\frac{\phi(u)-1-a_\phi(u-1)}{(u-1)^2}\right| < \infty.$$

Suppose that $\mathbb{E}\left[|\log a_\phi|^{2+\varepsilon}\right] < \infty$, $\mathbb{E}\left[|\log \beta_k^i|^{2+\varepsilon}\right] < \infty$, *for some* $\varepsilon > 0$, *all* i,k, *and that* $\mathbb{E}[\log a_\phi] = 0$. *Then the SDS on* $[0,1]$ *is conjugated to an asymptotically linear SDS on* $\mathbb{R}$ *that satisfy the hypotheses of Theorem 1.3. Therefore there exists at least one invariant Radon measure* $\widetilde{\nu}$ *on* $(0,1)$ *and for every such a measure* $\widetilde{\nu}$*, which charges a neighborhood of 0, there exists a strictly positive constant* C *such that for all* $0<a<b<1$

$$\lim_{z\to+\infty} \widetilde{\nu}(a/z, b/z) = C\log b/a$$

Proof. Let

$$r(u) = -\frac{1}{u} + \frac{1}{1-u}.$$

be a diffeomorphism of $(0,1)$ onto $\mathbb{R}$. In the technical Lemma A.4, whose proof is postponed to the Appendix A, we prove that the conjugated function $\Psi_\phi = r\circ\phi\circ r^{-1}$ satisfy (AL) for $A(\Psi_\phi) = 1/a_\phi$ and

$$B(\Psi_\phi) < C_r\left(\frac{(1+a_\phi+\beta_3^0)}{a_\phi\beta_2^0} + \frac{1}{\beta_1^0} + \frac{(1+a_\phi+\beta_3^1)}{a_\phi\beta_2^1} + \frac{1}{\beta_1^1}\right),$$

where C_r depends only on the function r. Thus, under the hypotheses of the corollary, the conjugated SDS satisfies the assumptions of our main theorem.

Let $\widetilde{\mu}$ be the law of ϕ and $\mu = r*\widetilde{\mu}*r^{-1}$ be the law of the conjugated SDS on $\mathbb{R}$. Then ν is a μ-invariant Radon measure on $\mathbb{R}$ if and only if $\widetilde{\nu} = r^{-1}*\nu$ is a $\widetilde{\mu}$-invariant Radon measure on $(0,1)$. Then by Theorem 1.3 and since $|r(u)+1/u| < 2$ for $0<u<1/2$,

$$\begin{aligned}\left|\widetilde{\nu}\Big(\frac{a}{z},\frac{b}{z}\Big) - \nu\Big(-\frac{z}{a},-\frac{z}{b}\Big)\right| &= \left|\nu\Big(r\Big(\frac{a}{z}\Big), r\Big(\frac{b}{z}\Big)\Big) - \nu\Big(-\frac{z}{a},-\frac{z}{b}\Big)\right| \\ &\le \nu\Big(-\frac{z}{a}-2, -\frac{z}{a}+2\Big) + \nu\Big(-\frac{z}{b}-2,-\frac{z}{b}+2\Big) \to 0\end{aligned}$$

for $z\to+\infty$. Thus

$$\lim_{z\to+\infty}\widetilde{\nu}\big(a/z, b/z\big) = \lim_{z\to+\infty}\nu\Big(-\frac{z}{a},-\frac{z}{b}\Big) = C\log b/a.$$

□

6.4. **Additive Markov processes and power functions.** When an asymptotically linear SDS is conjugated by a homeomorphism of the real line which behaves as the exponential at infinity, it is transformed into a SDS that is asymptotically a translation or, by the reversed conjugation, a power function.

More precisely consider a SDS generated by functions ϕ such that

$$|\phi(x) - x + \operatorname{sign}(x)u_\phi| \leq v_\phi \mathrm{e}^{-|x|} \tag{6.2}$$

for some constants u_ϕ and v_ϕ. This class contains mappings of $[0,\infty)$ that are equal to translations outside a bounded set, that is a Markov additive process as defined in Aldous [1, sections C11, C33]. Let s be a continuous bijection of $\mathbb{R}$ such that

$$s(x) = \mathrm{e}^x \text{ for } x > 1 \quad \text{and} \quad s(x) = -\mathrm{e}^{-x} \text{ for } x < -1.$$

Then the SDS generated by $\psi_\phi(x) = s \circ \phi \circ s^{-1}$ satisfies hypothesis (AL) with $A(\psi_\phi) = \mathrm{e}^{-u_\phi}$. Hence, under moment conditions that can be obtained with standard calculations, if $\mathbb{E}(u_\phi) = 0$ there exists an invariant measure which behaves at infinity as the Lebesgue measure dx, i.e.

$$\lim_{z\to+\infty} \widetilde{\nu}(\alpha + z, \beta + z) = C(\beta - \alpha),$$

for every measure of unbounded support, some constant $C > 0$ and all $\beta > \alpha$.

In a similar way a SDS generated by function ϕ such that

$$|x|^a \cdot \operatorname{sign}(x)\mathrm{e}^{-b_1 \log(|x|+2)^\alpha} \leq \phi(x) \leq |x|^a \cdot \operatorname{sign}(x)\mathrm{e}^{+b_1 \log(|x|+2)^\alpha},$$

for some α is associated to an α-asymptotically linear system by the reverse conjugation $\psi_\phi(x) = s^{-1} \circ \phi \circ s$ and $A(\psi_\phi) = a$. Thus, if $\mathbb{E}(\log a) = 0$ and some moments are finite, for any invariant measure $\widetilde{\nu}$, whose support in unbounded on the positive half-line, there exists a strictly positive constant C such that for all $1 < \alpha < \beta$

$$\lim_{z\to+\infty} \widetilde{\nu}(\alpha^z, \beta^z) = C \log \frac{\log \beta}{\log \alpha}.$$

6.5. **Population of Galton-Watson tree with random reproduction law.** Consider the following model of reproduction of a population. Let $\{\rho_\omega | \omega \in \Omega\}$ be the set of probability measures on the set of natural numbers $\mathbb{N}$ and $\lambda(d\omega)$ be a probability law on Ω. At each generation a law of reproduction ρ_ω is chosen according to $\lambda(d\omega)$ and each individual j is replaced by r_j offsprings, r_j chosen according to the law ρ_ω and independently from the other individuals. To prevent the extinction of the population a random immigration i_ω it added to the population. More formally, if the population consists of $x \in \mathbb{N}$ individuals, the population of the following generation is

$$\psi_{\omega,\mathbf{r}}(x) = i_\omega + \sum_{j=1}^{x} r_j,$$

where the reproduction law $\omega \in \Omega$ is chosen according to $\lambda(d\omega)$, $\mathbf{r} = \{r_j\}_j$ are i.i.d of law ρ_ω and i_ω is a random variable. If every generation is independent from the previous one then the evolution of the population is a SDS on $\mathcal{R} = \mathbb{N}$ of law $\mu(d\psi) = \otimes\rho_\omega(d\mathbf{r})\lambda(d\omega)$. If $\mathbb{E}r_1^2 < \infty$, the law of iterated logarithm proves that the $\psi_{\omega,\mathbf{r}}$ are μ-almost surely α-asymptotically linear with an error of order x^α for all $\alpha > 1/2$ and

$$A(\psi) = A_\omega = \int_{\mathbb{N}} r\rho_\omega(dr) = \text{average number of offspring per individual for } \rho_\omega \ .$$

Unlike the classical Galton-Watson process, in our context A_ω is not constant, but varies from one generation to another. The key parameter, that decides whether the system is recurrent, is $\mathbb{E}(\log A_\omega) = \int \log A_\omega \lambda(d\omega)$. To apply Theorem 1.3, we need to control moments of $B(\psi)$. The details are stated in the following lemma. Our estimates are fairly rough and the hypotheses could be probably improved, but this go beyond the purpose of our paper.

Lemma 6.3. *Suppose* $\mathbb{E}(r_1^4) = \int_\Omega \int_\mathbb{N} r_1^4 \rho_\omega(dr)\lambda(d\omega) < \infty$. *Let* $\alpha > 3/4$ *and*

$$B(\psi) = B_{\omega,\mathbf{r}} = \sup_{x\in\mathbb{N}} \frac{|\psi_{\omega,\mathbf{r}}(x) - A_\omega x|}{x^\alpha + 1},$$

then there exists a finite constant C_α *that only depends on* α *such that*

$$\mathbb{E}((\log^+ B(\psi))^{2+\epsilon}) \le C_\alpha \left(1 + \mathbb{E}((\log^+ i_\omega)^{2+\epsilon}) + \mathbb{E}(r_1^4)\right)$$

Proof. Observe first

$$\frac{|\psi_{\omega,\mathbf{r}}(x) - A_\omega x|}{x^\alpha + 1} \le i_\omega + \frac{\left|\sum_{j=1}^x (r_j - A_\omega)\right|}{x^\alpha + 1}.$$

Thus

$$(\log^+ B(\psi))^{2+\epsilon} \le C\left((C + \log^+ i_\omega)^{2+\epsilon} + \sup_{x\in\mathbb{N}} \left(\log^+ \frac{\left|\sum_{j=1}^x (r_j - A_\omega)\right|}{x^\alpha + 1}\right)^{2+\epsilon}\right)$$

Let $y_j := r_j - A_\omega$ be centered random variables. For a fixed reproduction law ω denote by $\mathbb{P}_\omega$ the quenched probability. Since under $\mathbb{P}_\omega$ the variables y_j are independent, $\mathbb{E}_\omega(y_{j_1}y_{j_2}y_{j_3}y_{j_4}) = 0$ if there exists an index j_k that is different from all the others. Then standard calculus shows that

$$\begin{aligned}
\mathbb{E}_\omega\left[\sum_{j=1}^x y_j\right]^4 &= \sum_{j_1,j_2,j_3,j_4=1}^x \mathbb{E}_\omega(y_{j_1}y_{j_2}y_{j_3}y_{j_4}) \\
&= x\mathbb{E}_\omega(y_1^4) + 3x(x-1)\big(\mathbb{E}_\omega[y_1^2]\big)^2 \le 4x^2\mathbb{E}_\omega(y_1^4)
\end{aligned}$$

Finally, since $\alpha > 3/4$, we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{x\in\mathbb{N}} \left(\log^+ \frac{\left|\sum_{j=1}^x (r_j - A_\omega)\right|}{x^\alpha + 1}\right)^{2+\epsilon}\right) &\le C\mathbb{E}\left(\sum_{x\in\mathbb{N}} \left(\frac{\left|\sum_{j=1}^x y_j\right|}{x^\alpha + 1}\right)^4\right) \\
&= C\mathbb{E}\left(\sum_{x\in\mathbb{N}} \frac{\mathbb{E}_\omega\left[\left|\sum_{j=1}^x y_j\right|^4\right]}{(x^\alpha + 1)^4}\right) \\
&\le C\mathbb{E}\left(\sum_{x\in\mathbb{N}} \frac{4x^2\mathbb{E}_\omega(y_1^4)}{(x^\alpha + 1)^4}\right) = C\sum_{x\in\mathbb{N}} \frac{4x^2\mathbb{E}(y_1^4)}{(x^\alpha + 1)^4} < \infty.
\end{aligned}$$

□

6.6. **Reflected random walk in critical case.** The reflected random walk

$$Y_n^x = \left|Y_{n-1}^x - u_n\right|,$$

is an example of an asymptotic translation for which (6.2) holds. Thus we can apply our main Theorem 1.3. However, in this case the same results hold under weaker hypotheses and a much more direct proof. We give here the proof of Theorem 1.11, stated in the introduction.

Proof of Theorem 1.11. Define the upward ladder times of $S_n = \sum_{i=1}^n u_i$:

$$\begin{aligned}
t_0 &= 0, \\
t_{k+1} &= \inf\{n > t_k : \ S_n \ge S_{t_k}\},
\end{aligned}$$

and put $\overline{u}_k = S_{t_k} - S_{t_{k-1}}$. Then $\{\overline{u}_k\}$ is a sequence of i.i.d. random variables and every $\overline{u}_k$ is equal in distribution to S_{t_1}. We define reflected random walk for $\{\overline{u}_k\}$:

$$\begin{aligned} \overline{Y}^x_0 &= x, \\ \overline{Y}^x_{k+1} &= \big|\overline{Y}^x_k - \overline{u}_{k+1}\big|, \end{aligned}$$

then $\overline{Y}^x_k = Y^x_{t_k}$. In view of the result of Chow and Lai [9], $\mathbb{E}(\overline{u}_k)^{\frac{1}{2}} < \infty$ and this is sufficient for existence of a unique invariant probability measure ν_t of the process $\{\overline{Y}^x_k\}$ (see [22] for more details). Let us define the measure

$$\nu_0(f) = \int_{\mathbb{R}_+} \mathbb{E}\Big[\sum_{n=0}^{t-1} f(Y^x_n)\Big]\nu_t(dx).$$

Notice first that this is a Radon measure. Indeed, define $l_i = \inf\{n > l_{i-1} :\ S_n < S_{l_{i-1}}\}$. Since $\mathbb{E}(u_1^-)^2 < \infty$, $-\infty < \mathbb{E}S_l < 0$ (see [9]). Take any $f \in C_C(\mathbb{R}_+)$, then by the duality Lemma [14]

$$\nu_0(f) = \int_{\mathbb{R}_+} \mathbb{E}\Big[\sum_{n=0}^{t-1} f(x - S_n)\Big]\nu_t(dx) = \int_{\mathbb{R}_+} \mathbb{E}\Big[\sum_{n=0}^{\infty} f(x - S_{l_n})\Big]\nu_t(dx). \tag{6.4}$$

By the renewal theorem

$$\mathbb{E}\Big[\sum_{n=0}^{\infty} f(x - S_{l_n})\Big] \quad\leq\quad C\mathbb{E}\big[\#n :\ \alpha < x - S_{l_n} < \beta\big] \leq C|\beta - \alpha|,$$

therefore $\nu_0(f)$ is finite and thus ν_0 is a Radon measure

Next, since $\mu_t * \nu_t = \nu_t$, we have

$$\begin{aligned} \mu * \nu_0(f) &= \int_{\mathbb{R}_+}\int_{\mathbb{R}} \mathbb{E}\Big[\sum_{n=0}^{t-1} f(Y^{|x-y|}_n)\Big]\mu(dy)\nu_t(dx) \\ &= \int_{\mathbb{R}_+} \mathbb{E}\Big[\sum_{n=1}^{t} f(Y^x_n)\Big]\nu_t(dx) = \nu_0(f). \end{aligned}$$

Therefore ν_0 is a μ invariant Radon measure, so $\nu_0 = C\nu$ and without any loss of generality we may assume $\nu = \nu_0$.

Finally, by (6.4), the Lebesgue theorem and the renewal theorem

$$\lim_{z\to\infty} \int_{\mathbb{R}_+} f(u - z)\nu(du) = \lim_{z\to\infty} \int_{\mathbb{R}_+} \mathbb{E}\Big[\sum_{n=0}^{\infty} f(x - S_{l_n} - z)\Big]\nu_t(dx) = \frac{1}{-\mathbb{E}S_l}\int_{\mathbb{R}_+} f(x)dx$$

and the Theorem is proved. □

Appendix A. Proofs of technical results

In this Appendix we give the postponed proofs of the technical results stated in Lemma 2.1, Proposition 4.1, Lemma 4.7. At the end we formulate and prove Lemma A.4, which is used in Section 6.3.

Proof of Lemma 2.1. We will prove the result only for positive x, since for negative values of x the same argument is valid just by conjugating with the map $x \mapsto -x$.

Suppose first $x \ge 1$. We have

$$\begin{aligned} r\left(A_\alpha r^{-1}(x) - B_\alpha(1+|r^{-1}(x)|^\alpha)\right) &\le \psi_r(x) \le \quad r\left(A_\alpha r^{-1}(x) + B_\alpha(1+|r^{-1}(x)|^\alpha)\right) \\ r\left(A_\alpha x^{\frac{1}{1-\alpha}} - B_\alpha(1+x^{\frac{\alpha}{1-\alpha}})\right) &\le \psi_r(x) \le \quad r\left(A_\alpha x^{\frac{1}{1-\alpha}} + B_\alpha(1+x^{\frac{\alpha}{1-\alpha}})\right) \\ r\left(A_\alpha x^{\frac{1}{1-\alpha}} - B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}}\right) &\le \psi_r(x) \le \quad r\left(A_\alpha x^{\frac{1}{1-\alpha}} + B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}}\right) \end{aligned}$$

where c_α only depends on α. Suppose further $x > c_\alpha B_\alpha / A_\alpha$, then $A_\alpha x^{\frac{1}{1-\alpha}} - B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}} > 0$ and

$$\begin{aligned} \left(A_\alpha x^{\frac{1}{1-\alpha}} - B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}}\right)^{1-\alpha} &\le \psi_r(x) \le \quad \left(A_\alpha x^{\frac{1}{1-\alpha}} + B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}}\right)^{1-\alpha} \\ A_\alpha^{1-\alpha} x^{\frac{1-\alpha}{1-\alpha}} - A_\alpha^{-\alpha} x^{\frac{-\alpha}{1-\alpha}} B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}} &\le \psi_r(x) \le \quad A_\alpha^{1-\alpha} x^{\frac{1-\alpha}{1-\alpha}} + (1-\alpha) A_\alpha^{-\alpha} x^{\frac{-\alpha}{1-\alpha}} B_\alpha c_\alpha x^{\frac{\alpha}{1-\alpha}} \\ A_\alpha^{1-\alpha} x - A_\alpha^{-\alpha} B_\alpha c_\alpha &\le \psi_r(x) \le \quad A_\alpha^{1-\alpha} x + (1-\alpha) A_\alpha^{-\alpha} B_\alpha c_\alpha \end{aligned}$$

since for $x_0 > 0$ and $h > 0$, by concavity $(x_0+h)^{1-\alpha} \le x_0^{1-\alpha} + (1-\alpha)x_0^{-\alpha} h$ and $(x_0-h)^{1-\alpha} \ge x_0^{1-\alpha} - x_0^{-\alpha} h$. Hence we proved the lemma for $x > \max\{1, c_\alpha B_\alpha/A_\alpha\}$ Now, for $x < 1$

$$\begin{aligned} r\left(A_\alpha x^{\frac{1}{1-\alpha}} - B_\alpha(1+x^{\frac{\alpha}{1-\alpha}})\right) &\le \psi_r(x) \le \quad r\left(A_\alpha x^{\frac{1}{1-\alpha}} + B_\alpha(1+x^{\frac{\alpha}{1-\alpha}})\right) \\ -(2B_\alpha)^{1-\alpha} &\le \psi_r(x) \le \quad (A_\alpha + 2B_\alpha)^{1-\alpha} \end{aligned}$$

and for $x \le c_\alpha B_\alpha / A_\alpha$.

$$-\left(B_\alpha\left(1+\left(\frac{c_\alpha B_\alpha}{A_\alpha}\right)^{\frac{\alpha}{1-\alpha}}\right)\right)^{1-\alpha} \le \psi_r(x) \le \quad \left(A_\alpha\left(\frac{c_\alpha B_\alpha}{A_\alpha}\right)^{\frac{1}{1-\alpha}} + B_\alpha\left(1+\left(\frac{c_\alpha B_\alpha}{A_\alpha}\right)^{\frac{\alpha}{1-\alpha}}\right)\right)^{\frac{\alpha}{1-\alpha}}$$

Hence the lemma follows. □

Proof of Proposition 4.1. **Step 1.** Let t_k and l_k be the stopping times defined in (2.7). Let U_l be the potential of the random walk S_{l_k} and let

$$R(x) := \sum_{k=0}^{\infty} \mathbb{E}\big[g(x+S_{l_k})\big] = U_l(\delta_x *_{\mathbb{R}} g).$$

Since the function g is directly Riemann integrable and $-\infty < \mathbb{E}S_l < 0$, the function R is well defined and finite for every x. Notice also that by the duality lemma [14]

$$R(x) = \sum_{k=0}^{\infty} \mathbb{E}\big[g(x+S_{l_k})\big] = \mathbb{E}\Big[\sum_{k=0}^{t-1} g(x+S_k)\Big]. \tag{A.1}$$

Step 2. We claim that

$$\mathbb{E}\big[f(x+S_t)\big] - f(x) = \mathbb{E}\left[\sum_{k=0}^{t-1} g(x+S_k)\right] = R(x). \tag{A.2}$$

Indeed, the process $f(x+S_n) - \sum_{k=0}^{n-1} g(x+S_k)$ forms a martingale (for this purpose one just has to iterate the Poisson equation (4.3)). Then for any fixed n, $T \wedge n$ is a bounded stopping time, therefore by the optional stopping time theorem we have

$$f(x) = \mathbb{E}\Big[f(x+S_{t\wedge n})\Big] - \mathbb{E}\left[\sum_{k=0}^{(t\wedge n)-1} g(x+S_k)\right].$$

To justify that we can let n tend to infinity and change the order of the limit and the expected value to obtain (A.2) observe that

$$\mathbb{E}\big[f(x+S_{t\wedge n})\big] \le C\mathbb{E}\big[1+(x+S_{t\wedge n})^+\big] \le C\mathbb{E}\big[1+(x+S_t)^+\big] < \infty.$$

The second term is uniformly dominated in n by

$$\mathbb{E}\Big[\sum_{k=0}^{t-1}|g|(x+S_k)\Big] = \sum_{k=0}^{\infty}\mathbb{E}\big[|g|(x+S_{l_k})\big] < \infty,$$

therefore converges to $R(x)$ when n goes to infinity.

This proves that

$$\mathbb{E}\big[f(x+S_t)\big] - f(x) = R(x) = U_l(\delta_x *_{\mathbb{R}} g)$$

and by the renewal theorem we obtain (4.4).

Step 3. Let

$$G(x) := \int_{-\infty}^{x} g(z)dz.$$

If we suppose $\int g(x)dx = 0$ then

$$G(x) = \int_{-\infty}^{+\infty} g(z)dz - \int_{x}^{\infty} g(z)dz = -\int_{x}^{+\infty} g(z)dz.$$

Thus

$$|G(x)| \le \int_{-\infty}^{x}|g(z)|dz\,\mathbf{1}_{(-\infty,0]}(x) + \int_{x}^{\infty}|g(z)|dz\,\mathbf{1}_{[0,+\infty)}(x) =: G_1(x) + G_2(x),$$

and G is directly Riemann integrable since functions G_i are monotone and integrable on their support:

$$\begin{aligned}\int_{-\infty}^{0} G_1(x)dx &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{1}_{[z<x<0]}|g(z)|dx\,dz = \int_{-\infty}^{0}|zg(z)|\,dz < \infty\\ \int_{0}^{+\infty} G_2(x)dx &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{1}_{[z>x>0]}|g(z)|dx\,dz = \int_{0}^{+\infty}|zg(z)|\,dz < \infty.\end{aligned}$$

Furthermore

$$\begin{aligned}\int_{-\infty}^{\infty} G(x)dx &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{1}_{[z<x<0]}g(z)dx\,dz - \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathbf{1}_{[z>x>0]}g(z)dx\,dz\\ &= -\int_{-\infty}^{+\infty} zg(z)dz.\end{aligned}$$

Step 4. By the renewal theory, $U_l(\delta_x *_{\mathbb{R}} G)$ is well defined and by Fubini's thoerem

$$\int_{-\infty}^{x} R(z)dz = \int_{-\infty}^{0}\int_{-\infty}^{x} g(z+u)dzU_l(du) = \int_{-\infty}^{0}\int_{-\infty}^{x+u} g(z)dzU_l(du) = U_l(\delta_x *_{\mathbb{R}} G).$$

On the other hand

$$\int_{-\infty}^{x} R(z)dz = \mathbb{E}\Big[\int_{-\infty}^{x} f(z+S_t)dz - \int_{-\infty}^{x} f(z)dz\Big] = \mathbb{E}\Big[\int_{x}^{x+S_t} f(z)dz\Big].$$

In fact the two integrals above are finite because by our hypotheses

$$\int_{-\infty}^{x}\mathbb{E}[f(y+S_t)]dy = \mathbb{E}\Big[\int_{-\infty}^{x+S_t} f(y)dy\Big] \le C\mathbb{E}[1+(x+S_t)^+]$$

and $\mathbb{E}S_t < \infty$ since $\overline{\mu}$ has moment of order $2+\epsilon$, see [9]. Thus we proved

$$\mathbb{E}\Big[\int_x^{x+S_t} f(z)dz\Big] = \delta_x *_{\mathbb{R}} U_l(G)$$

and we can conclude using again the renewal theorem. □

Proof of Lemma 4.7. **Step 1.** Let $0 < \gamma < 1$, then the set of $v > 0$ such that the function $u \mapsto (u-v)^{-\gamma}$ is ν-integrable on $(v, +\infty)$ is of full Lebesgue measure. In fact for any interval $[a, b] \subset (0, \infty)$:

$$\begin{aligned}
\int_a^b \left(\int_v^\infty (u-v)^{-\gamma}\nu(du)\right) dv &= \int_a^{2b} \left(\int_a^{u\wedge b} (u-v)^{-\gamma} dv\right) \nu(du) + \int_{2b}^\infty \left(\int_a^{u\wedge b} (u-v)^{-\gamma} dv\right) \nu(du) \\
&\le \int_a^{2b} \left(\int_0^{2b-a} w^{-\gamma} dw\right) \nu(du) + \int_{2b}^\infty \left(\int_{u-b}^{u-a} w^{-\gamma} dw\right) \nu(du) \\
&= C + \int_{2b}^\infty (u-b)^{-\gamma}(b-a)\nu(du) < \infty
\end{aligned}$$

Take w_0 such that $\int_{w_0}^\infty (u-w_0)^{-\gamma}\nu(du) < \infty$ then

$$f_\phi(x) = \int_{w_0}^\infty \phi(e^{-x}(u-w_0))\nu(du) \le C\int_{w_0}^\infty e^{\gamma x}(u-w_0)^{-\gamma}\nu(du) \le Ce^{\gamma x},$$

this gives good estimates of f_ϕ for negative x's.

Step 2. Let $\mathrm{supp}(\phi) \subset [m, M]$. By Proposition 3.1 the tail of ν is at most logarithmic, therefore for $x \ge 0$,

$$f_\phi(x) \le \nu([e^x m + w_0, e^x M + w_0]) \le \nu([e^x m, e^x(M + w_0)]) \le C(1 + x).$$

and

$$\begin{aligned}
\int_{-\infty}^x f_\phi(y)dy &\le C\int_{\mathbb{R}}\int_{-\infty}^\infty \mathbf{1}_{[y<x]}\mathbf{1}_{[m,M]}(\mathrm{e}^{-y}(u-w_0))dy\nu(du) \\
&\le C\int_{\mathbb{R}} \mathbf{1}_{[w_0<u\le \mathrm{e}^x(M+w_0)]}\log\frac{M}{m}\nu(du) \le C(1+x^+)
\end{aligned}$$

This proves (4.2).

Step 3. We need to justify that $g_\phi = \overline{\mu} * f_\phi - f_\phi$ is directly Riemann integrable and moreover $\int_{\mathbb{R}} |xg(x)|dx < \infty$. We recall first that, since g is continuous, to prove that it is directly Riemann integrable is sufficient to show that $|g|$ is dominated on $(-\infty, 0]$ (resp. on $[0, +\infty)$) by an integrable nondecreasing (resp. nonincreasing) function. For $x < 0$:

$$\begin{aligned}
\overline{\mu} * f_\phi(x) &= \int_{-\infty}^{+\infty} f_\phi(x+y)\overline{\mu}(dy) \\
&= \int_{-\infty}^{-x/2} C\mathrm{e}^{\gamma(x+y)}\overline{\mu}(dy) + \int_{-x/2}^{+\infty} K(1+(x+y)^+)\overline{\mu}(dy) \\
&\le C\mathrm{e}^{\gamma(x/2)} + \int_{-x/2}^\infty K(1+|y|)\overline{\mu}(dy) = C\mathrm{e}^{\gamma(x/2)} + \frac{1}{|x|^{2+\varepsilon}}\int_{-x/2}^\infty K(1+|y|)|y|^{1+\varepsilon}\overline{\mu}(dy) \\
&\le \frac{C}{1+|x|^{1+\varepsilon}},
\end{aligned}$$

since $\overline{\mu}$ has a moment of order $2+\varepsilon$. Thus $g_\phi \mathbf{1}_{(-\infty,0]}$ is directly Riemann integrable. Furthermore

$$\begin{aligned}
\int_{-\infty}^{0} |x|\overline{\mu} * f_\phi(x)dx &= \int_{-\infty}^{+\infty}\int_{-\infty}^{0} |x| f_\phi(x+y)dx\overline{\mu}(dy)\\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{y} |x-y| f_\phi(x)dx\overline{\mu}(dy)\\
&= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{0} |x-y| f_\phi(x)dx + \int_{0}^{y^+} |x-y| f_\phi(x)dx\right)\overline{\mu}(dy)\\
&\le \int_{-\infty}^{+\infty}\left(C\int_{-\infty}^{0} |x-y| \mathrm{e}^{\gamma x}dx + 2|y|\int_{0}^{y^+} f_\phi(x)dx\right)\overline{\mu}(dy)\\
&\le C\int_{-\infty}^{+\infty}(1+|y|+|y|^2)\overline{\mu}(dy) < \infty
\end{aligned}$$

Step 4. To check that g_ϕ is directly Riemann integrable and $|xg_\phi(x)|$ is integrable for positive x we show that:

$$\sum_{n=0}^{\infty}\sup_{n\le x<n+1}|xg_\phi(x)|<\infty.$$

Applying μ_0 invariance of ν_0 and since $A(\psi_0)=A(\psi)$, we obtain

$$|g(x)| = \Big|\int\int \phi\big(e^{-x}(A(\psi)u)\big) - \phi\big(e^{-x}(\psi(u))\big)\nu_0(du)\mu_0(d\psi)\Big|$$

The function $\widetilde{\phi}(x)=\phi(\mathrm{e}^x)$ is a Lipschitz on $\mathbb{R}$, hence:

$$\begin{aligned}
\Big|\phi\big(e^{-x}(A(\psi)u)\big) - \phi\big(e^{-x}\psi(u)\big)\Big| &\le \min\big\{C\big|\log\frac{\psi(u)}{A(\psi)u}\big|, 2\|\phi\|_\infty\big\}\\
&\le \min\big\{C\big|\frac{\psi(u)}{A(\psi)u}-1\big|, 2\|\phi\|_\infty\big\}\\
&\le \min\big\{C\big|\frac{B(\psi)}{A(\psi)u}\big|, 2\|\phi\|_\infty\big\} =: \rho(\frac{Au}{B})
\end{aligned}$$

where we use the convention that $\log z=-\infty$ for $z\le 0$ and $\rho(y):=\min\{C\big|\frac{1}{y}\big|, 2\|\phi\|_\infty\}$. Take now $0\le n\le x<n+1$

$$\begin{aligned}
&|x|\Big|\phi\big(e^{-x}(Au)\big) - \phi\big(e^{-x}\psi(u)\big)\Big|\\
&\qquad\le \log^+\frac{Au+B}{m}\cdot\rho(\frac{Au}{B})\left(\mathbf{1}_{[\log\frac{\psi(u)}{M\mathrm{e}}\le n\le \log\frac{\psi(u)}{m}]} + \mathbf{1}_{[\log\frac{Au}{M\mathrm{e}}\le n\le\log\frac{Au}{m}]}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{n=0}^{\infty}\sup_{n\le x<n+1}|xg_\phi(x)| &\le \int\int\sum_{n=0}^{\infty}\sup_{n\le x<n+1}|x|\Big|\phi\big(e^{-x}(Au)\big)-\phi\big(e^{-x}\psi(u)\big)\Big|\nu_0(du)\mu_0(d\psi)\\
&\le \int\int \log^+\frac{Au+B}{m}\cdot\rho(\frac{Au}{B})2\log\frac{\mathrm{e}M}{m}\nu_0(du)\mu_0(d\psi)\\
&\le 2\log\frac{\mathrm{e}M}{m}\int\Big(\int\Big(\log^+\frac{1}{m}+\log^+ B+\log^+(\frac{Au}{B}+1)\Big)\rho(\frac{Au}{B})\nu_0(du)\Big)\mu_0(d\psi)
\end{aligned}$$

To estimate the last expression we use the fact that there exists a constant C such that for all non-increasing functions $h : [0, +\infty) \to [0, +\infty)$ and all $M > 0$

$$\int_{\mathbb{R}} h\big(|u|/M\big)\nu_0(du) \le C(1+\log^+ M)\bigg(\|h\|_\infty + \int_1^{+\infty} h(z)(1+\log(z))\frac{dz}{z}\bigg). \tag{A.3}$$

Before we prove the last inequality, let us check how it implies the lemma. Since $\log^+(z+1)\rho(z) \le C/(1+z)^{1/2}$ for $z > 0$, by (A.3), we have

$$\begin{aligned}
&\int \bigg(\log^+\frac{1}{m} + \log^+ B + \log^+(\frac{Au}{B}+1)\bigg)\,\rho(\frac{Au}{B})\nu_0(du)\\
&\le C(1+\log^+(B/A))\bigg((1+\log^+ B) + \int_1^{+\infty}\bigg((1+\log^+ B)\rho(z) + \frac{1}{(1+z)^{1/2}}\bigg)(1+\log^+(z))\frac{dz}{z}\bigg)\\
&\qquad\qquad \le C\left(1 + (\log^+ B)^2 + \log^+ B\log^+ A\right).
\end{aligned}$$

The last expression is μ_0-integrable and we conclude.

Finally to prove (A.3) we write

$$\begin{aligned}
\int_{\mathbb{R}} h\big(|u|/M\big)\nu_0(du) &\le \|h\|_\infty\nu_0([-M\mathrm{e}, M\mathrm{e}]) + \int \mathbf{1}_{[|u|>\mathrm{e}M]} h\big(|u|/M\big)\nu_0(du)\\
&\le C(1+\log^+ M)\|h\|_\infty + \sum_{n=1}^{\infty}\int \mathbf{1}_{[\mathrm{e}^{n+1}M \ge |u| > \mathrm{e}^n M]}\nu_0(du)h(e^n)\\
&\le C(1+\log^+ M)\|h\|_\infty + \sum_{n=1}^{\infty}\big(\log^+(\mathrm{e}^{n+1}M)+1\big)h(e^n)\\
&\le C(1+\log^+ M)\|h\|_\infty + \sum_{n=1}^{\infty}\int_{\mathrm{e}^{n-1}}^{\mathrm{e}^n}\big(\log^+(z\mathrm{e}^2 M)+1\big)h(z)\frac{dz}{z}\\
&\le C(1+\log^+ M)\|h\|_\infty + \int_1^{\infty}\big(\log^+(z) + \log^+ M + 3\big)h(z)\frac{dz}{z}.
\end{aligned}$$

□

Lemma A.4. *Let $\phi \in C([0,1])$ be a function fixing 0 and 1, derivable at 0 and 1 and such that $\phi'(0) = \phi'(1) =: a_\phi$. Suppose:*

$$\beta_1^0 = \inf_{u\in[0,1/2]}(1-\phi(u)) > 0, \quad \beta_2^0 = \inf_{u\in[0,1/2]}\frac{\phi(u)}{u} > 0, \quad \beta_3^0 = \sup_{u\in[0,1/2]}\left|\frac{\phi(u)-a_\phi u}{u^2}\right| < \infty.$$

$$\beta_1^1 = \inf_{u\in[1/2,1]}\phi(u) > 0, \; \beta_2^1 = \inf_{u\in[1/2,1]}\frac{1-\phi(u)}{1-u} > 0, \; \beta_3^1 = \sup_{u\in[1/2,1]}\left|\frac{\phi(u)-1-a_\phi(u-1)}{(u-1)^2}\right| < \infty.$$

Consider the diffeomorphism of $(0,1)$ on $\mathbb{R}$

$$r(u) = -\frac{1}{u} + \frac{1}{1-u}.$$

Then $\Psi_\phi = r\circ\phi\circ r^{-1}$ satisfy (AL) for $A(\Psi_\phi) = 1/a_\phi$ and

$$B(\Psi_\phi) < C_r\left(\frac{(1+a_\phi+\beta_3^0)}{a_\phi\beta_2^0} + \frac{1}{\beta_1^0} + \frac{(1+a_\phi+\beta_3^1)}{a_\phi\beta_2^1} + \frac{1}{\beta_1^1}\right),$$

where C_r depends only on the function r.

Proof. Since the function r satisfies $r(u) = -r(1-u)$ and our assumptions on ϕ near 0 and 1 are symmetric, it is sufficient to prove the condition (AL) only for negative x. Since $\beta_3^0 < \infty$, by the Taylor expansion we have

$$\phi(u) = au + \epsilon_\phi(u) \text{ with } |\epsilon_\phi(u)| \le \beta_3^0 u^2 \text{ for } u \le 1/2. \tag{A.5}$$

Moreover simple calculus shows that

$$r^{-1}(x) = -\frac{1}{x} + \epsilon_{r^{-1}}(x) \text{ with } \epsilon_{r^{-1}}(x) = O\left(\frac{1}{x^2}\right) \text{ for } x \to -\infty \tag{A.6}$$

For $x < 0$ we write

$$\left|\frac{x}{a_\phi} - \Psi_\phi(x)\right| = \left|\frac{x}{a_\phi} - r\big(\phi\big(r^{-1}(x)\big)\big)\right| \le \left|\frac{x}{a_\phi} + \frac{1}{\phi\big(r^{-1}(x)\big)}\right| + \frac{1}{1-\phi\big(r^{-1}(x)\big)}$$

Notice that for $x < 0$, $r^{-1}(x) \in (0, 1/2)$, therefore the second factor can be bounded by $\frac{1}{\beta_1^0}$. So, we need just to estimate the first term. We write

$$I(x) = \left|\frac{x}{a_\phi} - \frac{1}{\phi\big(r^{-1}(x)\big)}\right| = \frac{|\phi(r^{-1}(x))x - a_\phi|}{|a_\phi \cdot \phi(r^{-1}(x))|}$$

Take $M = -r(1/10)$, then for $x \in [-M, 0]$ we have $\phi(r^{-1}(x)) \ge \beta_2^0 r^{-1}(x) \ge \beta_2^0/10$ and hence

$$I(x) \le 10\frac{M + a_\phi}{a_\phi \beta_2^0}.$$

Now we consider $x < -M$. Since there exists η such that $xr^{-1}(x) > \eta$, by (A.5) and (A.6), we have

$$\begin{aligned} I(x) &= \frac{\big|\phi(r^{-1}(x))x + a_\phi\big| \cdot |x|}{a_\phi \cdot \frac{\phi(r^{-1}(x))}{r^{-1}(x)} \cdot |xr^{-1}(x)|} \le \frac{1}{a_\phi \beta_2^0 \eta} \cdot \big|\phi(r^{-1}(x))x + a_\phi\big||x| \\ &= \frac{1}{a_\phi \beta_2^0 \eta}\big|a_\phi r^{-1}(x)\, x + \varepsilon_\phi(r^{-1}(x))\; x + a_\phi\big||x| = \frac{1}{a_\phi \beta_2^0 \eta}\big|a_\phi \varepsilon_{r^{-1}}(x) x + \varepsilon_\phi\left(r^{-1}(x)\right) x\big| \\ &\le \frac{\big|a_\phi \varepsilon_{r^{-1}}(x)x\big| + \beta_3^0\big|\left(r^{-1}(x)\right)^2 x\big|}{a_\phi \beta_2^0 \eta} \le \frac{C_r(a_\phi + \beta_3^0)}{a_\phi \beta_2^0}. \end{aligned}$$

□

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