ON SOLUTIONS OF THE AFFINE RECURSION AND THE SMOOTHING TRANSFORM IN THE CRITICAL CASE

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ABSTRACT. In this paper we present a new result concerning description of asymptotics of the invariant measure of the affine recursion in the critical case. We discuss also relation of this model with the smoothing transform.

1. THE AFFINE RECURSION

We consider the random difference equation:

$$X =_d AX + B, \tag{1.1}$$

where $(A, B) \in \mathbb{R}^+ \times \mathbb{R}$ and X are independent random variables. This equation appears both in numerous applications outside mathematics (in economy, physics, biology) and in purely theoretical problems in other branches of mathematics. It is used to study e.g. some aspects of financial mathematics, fractals, random walks in random environment, branching processes, Poisson and Martin boundaries.

It is well known that if $\mathbb{E}[\log A] < 0$ and $\mathbb{E}[\log^+ |B|] < \infty$, then there exists a unique solution to (1.1). The solution is the limit in distribution of the Markov chain

$$\begin{aligned} X_0^x &= 0, \\ X_n^x &= A_n X_{n-1}^x + B_n, \end{aligned} \tag{1.2}$$

which is called the affine recursion (since the formula reflects the action of (A_n, B_n), an element of the affine group, on the real line). To simplify our notation we will write $X_n = X_n^0$.

The most celebrated result is due to Kesten [16] (see also Goldie [11]), who proved that if $\mathbb{E}A^\alpha = 1$ for some $\alpha > 0$ (and some other assumptions are satisfied), then

$$\lim_{t\to\infty} t^\alpha \mathbb{P}[|X| > t] = C_+,$$

i.e. if ν is the law of X, then $\nu(dx) \sim \frac{dx}{x^{1+\alpha}}$ at infinity.

We are interested here in the critical case, when $\mathbb{E}\log A = 0$. Then, equation (1.1) has no stochastic solutions. Nevertheless the equation can be written in terms of measures:

$$\mu * \nu = \nu, \tag{1.3}$$

where μ is the distribution of (A, B), and $\mu * \nu$ is defined as follows

$$\mu * \nu(f) = \int\int f(ax + b)\nu(dx)\mu(da, db).$$

In the nineties Babillot, Bougerol, Elie, [4] proved that that under the following hypotheses

$$\mathbb{E}\big[(|\log A| + \log^+ |B|)^{2+\varepsilon}\big] < \infty,\ \mathbb{P}[Ax + B = x] < 1 \text{ for all } x \in \mathbb{R} \ \text{ and } \mathbb{P}[A_1 = 1] < 1 \tag{1.4}$$

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there exists a unique (up to a constant factor) Radon measure ν, which is a solution to (1.3). The measure ν is an invariant measure of the process (1.2).

Recently we studied behavior of ν at infinity and we proved that for any $\alpha > \beta > 0$,

$$\lim_{x\to\infty} \nu(\beta x, \alpha x) = C_+,$$

for some strictly positive C_+, [5, 6]. In other words we proved that the measure ν behaves at infinity like $C_+ \frac{da}{a}$. Unfortunately this result was proved under very strong hypotheses. We assumed that exponential moments are finite, i.e.

$$\mathbb{E}\big[A^\delta + A^{-\delta} + |B|^\delta\big] < \infty \qquad \text{for some } \delta > 0, \tag{1.5}$$

moreover in [6] we needed also absolute continuity of the measure $\overline{\mu}$, the law of $\log A$.

In this paper we consider the affine recursion, when B is strictly positive, that implies also that the support of ν must be contained in $(0,\infty)$. It turns out, in these settings the assumptions can be weakened and exponential moments are not really needed. We restrict ourself to the aperiodic case, i.e. we assume the law of $\log A$ is not contained in a set of the form $p\mathbb{Z}$ for some positive p. Our main result is the following

Theorem 1.6. *Assume that* (1.4) *is satisfied, the measure* $\overline{\mu}$ *is aperiodic and the following holds*

$$\mathbb{E}\big[(|\log A| + \log^+ B)^{4+\varepsilon}\big] < \infty, \tag{1.7}$$

and

$$\mathbb{E}\big[|\log B|\big] < \infty, \qquad B \geq 0, \ \ a.s. \tag{1.8}$$

Then for every function $\phi \in C_c(\mathbb{R}^+)$

$$\lim_{z\to+\infty} \int_{\mathbb{R}^+} \phi(uz^{-1})\nu(du) = C_+ \int_{\mathbb{R}^+} \phi(a)\frac{da}{a}$$

for some strictly positive constant C_+.

Moreover for every $\alpha < \beta$

$$\lim_{z\to\infty} \nu\big(u: \ \alpha z < u < \beta z\big) = C_+ \log\frac{\beta}{\alpha}. \tag{1.9}$$

Notice that comparing with the main result of [5] we replace requirements of exponential moments (1.5) by much weaker assumption (1.7) and we assume additionally positivity of B. The integral condition in (1.7) is needed to control behavior of B and of the invariant measure close to 0, and it is unnecessary if $B > \delta$ a.s. for some $\delta > 0$.

A complete proof of this result will be given in section 3. The idea is the following. First one has to find some preliminary estimates of the measure ν under the hypothesis (1.4). Here we will just deduce from results contained in [5], that there exists a slowly varying function $L(z)$ such that the family of measures $\frac{\delta_{z^{-1}}*\nu}{L(z)}$ converges weakly to $C\frac{da}{a}$, i.e. the measure $\nu(da)$ behaves at infinity like $L(a)\frac{da}{a}$ (Proposition 3.1). Next applying the duality lemma, thanks to positivity of B, we prove that the measure ν is indeed bounded by the logarithm, more precisely we will show $\nu(0,z) \leq C(1+\log z)$ (Proposition (3.5)). Finally for an arbitrary compactly supported function ϕ on $\mathbb{R}^+$ we consider the function

$$f_\phi(x) = \int_{\mathbb{R}^+} \phi(ue^{-x})\nu(du)$$

as a solution of the Poisson equation

$$\overline{\mu} *_{\mathbb{R}} f_\phi = f_\phi + \psi_\phi \tag{1.10}$$

where ψ_ϕ is defined by the formula above, i.e. $\psi_\phi = \overline{\mu} *_{\mathbb{R}} f_\phi - f_\phi$. Then knowing already some estimates of the function f_ϕ (and our preliminary estimates are sufficient for that purpose) one can describe its asymptotics. There are two different methods. One bases on the classical results of Port and Stone [23, 24], who just solved explicitly the Poisson equation in the case when $\overline{\mu}$ is absolutely continuous. Nevertheless for our purpose much less is needed and the appropriate argument was given in [5]. The second method was introduced by Durrett and Liggett [10]. Thanks to the duality lemma they reduce the Poisson equation to the classical renewal equation, i.e. to an equation of the form (1.10), but with $\overline{\mu}$ replaced by measure with drift and ψ_ϕ replaced by some other function. In order to prove Theorem 1.6 we follow here the arguments given in [5]. The second method in the context of the affine recursion was considered by Kolesko [22] and in more general settings of Lipschitz recursions will be the subject of our other paper.

2. The smoothing transform

The measure ν described in Theorem 1.6 is not a probability measure, but only a Radon measure. However it turns out that this result has some applications in study purely probability objects. Here we will shortly present how this result and the methods can be used to study the smoothing transform.

To define the (inhomogeneous) smoothing transform take $(B, A_1, A_2, ...)$ to be a sequence of positive random variables and let N be a random natural number. On the set $P(\mathbb{R})$ of probability measures on the real line the smoothing transform is defined as follows

$$\mu \mapsto \mathcal{L}\Big(\sum_{j=1}^{N} A_j X_j + B\Big),$$

where $X_1, X_2, ...$ is a sequence of i.i.d random variables with common distribution μ, independent of $(B, A_1, A_2, ..)$ and N. $\mathcal{L}(X)$ denotes the law of the random variable X. A fixed point of the smoothing transform is given by any $\mu \in P(\mathbb{R})$ such that, if X has distribution μ, the equation

$$X =_d \sum_{j=1}^{N} A_j X_j + B, \tag{2.1}$$

holds true. Notice that if N and A_i, B are constants, the equation above characterizes stable laws as a particular case of (2.1).

We are interested also in a more specific case of (2.1). Taking $B = 0$ we obtain the homogeneous smoothing transform, i.e.

$$X =_d \sum_{i=1}^{N} A_i X_i. \tag{2.2}$$

Both stochastic equations described above are important from the point of view of applications. Equation (2.2) plays it role in description of e.g. interacting particle systems [10] and the branching random walk [13, 1]. In recent years, from very practical reasons, the inhomogeneous equation has gained importance. This equation appears e.g. in the stochastic analysis of the Pagerank algorithm (which in the heart of the Google engine) [14, 15] as well as in the analysis of a large class of divide and conquer algorithms including the Quicksort algorithm [19, 20].

Although (2.1) and (2.2) look similar to (1.1), often they turn out to have completely different properties. While studying equations (2.1) and (2.2) main concern is to describe the right hypotheses for the following issues: existence of solutions, characterization of all the solutions and finally, description of their properties.

2.1. **Homogeneous smoothing transform.** We start first with description of the homogeneous smoothing transform. The fixed points of equation (2.2) are governed by the convex function

$$m(\theta) = \mathbb{E}\Big[\sum_{j=1}^{N} A_j^\theta\Big]. \tag{2.3}$$

To exclude the trivial case we make the assumption $\mathbb{E}N > 1$. The first question that can be asked here is about existence of solutions of (2.2) and if there are any, what are all of them. The most important results are contained in the work of Durrett, Liggett [10] and in a series of papers of Liu e.g. [17]. They proved that the set of solutions of (2.2) is nonempty if and only if there is $\alpha \le 1$ such that $m(\alpha) = 1$ and $m'(\alpha) \le 0$. Moreover the parameter α describes the asymptotic of the Laplace transform of solutions. Their proofs goes via the Poisson equation as described in the previous section (of course some additional assumptions are needed). All their results are formulated in terms of the Laplace transform, but applying the Tauberian theorem for $\alpha < 1$ they give the correct asymptotics of X, a solution of (2.2). Namely they imply

$$\lim_{t\to\infty} t^\alpha \mathbb{P}[X > t] = C_1 \text{ if } m'(\alpha) < 0 \qquad \text{and} \qquad \lim_{t\to\infty} \frac{t^\alpha}{\log t}\mathbb{P}[X > t] = C_2 \text{ if } m'(\alpha) = 0.$$

Unfortunately the Tauberian theorem does not give the optimal answer when $\alpha = 1$ e.g. if $m'(\alpha) = 0$ one can deduce that

$$\int_0^x \mathbb{P}[X > t]dt \sim C_2 \log x \qquad \text{as } x \to \infty.$$

Thus, the results of [10, 17] are sharp only for $\alpha < 1$.

It turns out that to study the case $\alpha = 1$ one has to reduce the problem to the random difference equation (1.1). For reader's convenience we sketch here the arguments due to Guivarc'h [12], which work in the case when N is constant and A_i are i.i.d. For the general case see [18, 7].

Let X be a solution to (2.2). We introduce probability measures: let η be the law of X, θ the law of $\sum_{i=2}^N A_i X_i$, ρ the law of A. We define new measures: $\nu(dx) = x\eta(dx)$, $\widetilde{\rho}(da) = a\rho(da)$. Then, it turns out that the measure ν is μ invariant for $\mu(da\,db) = N\widetilde{\rho}(da)\otimes\theta(db)$ defined on $\mathbb{R}^+\times\mathbb{R}^+$, i.e. μ and ν satisfy (1.3). Indeed for any compactly supported function on $\mathbb{R}^+$ we have

$$\begin{aligned}
\nu(f) &= \int_{\mathbb{R}^+} f(x)\nu(dx) = \int_{\mathbb{R}^+} f(x)x\eta(dx) = \mathbb{E}\big[f(X)X\big]\\
&= \mathbb{E}\Big[f\Big(\sum_{i=1}^N A_iX_i\Big)\sum_{i=1}^N A_iX_i\Big] = N\mathbb{E}\Big[f\Big(A_1X_1 + \sum_{i=2}^N A_iX_i\Big)A_1X_1\Big]\\
&= N\int\int\int f(ax+b)ax\rho(da)\eta(dx)\theta(db) = \int\int f(ax+b)\big(N\widetilde{\rho}(da)\otimes\theta(db)\big)\nu(dx)\\
&= \int\int f(ax+b)\mu(da\,db)\nu(dx)
\end{aligned}$$

Assume now that $m(1) = 1$, $m'(1) < 0$ and there exists $\beta > 1$ such that $m(\beta) = 1$. Then observe, that μ is a probability measure and moreover

$$\begin{aligned}
\int \log a\mu(da\,db) &= N\int \log a\widetilde{\rho}(da) = N\int a\log a\rho(da) = m'(1) < 0,\\
\int a^{\beta-1}\mu(da\,db) &= N\int a^\beta\rho(da) = m(\beta) = 1.
\end{aligned}$$

One can easily check also other assumptions of the Kesten theorem, thus $\nu(dx) \sim C_+\frac{dx}{x^{(\beta-1)+1}} = C_+\frac{dx}{x^\beta}$, $\eta(dx) \sim C_+\frac{dx}{x^{\beta+1}}$ and finally $P[X > t] \sim C_+t^{-\beta}$ (we refer to [12, 18] for all the details).

Exactly the same argument is valid in the critical case when $m(1) = 1$ and $m'(1) = 0$. In fact this is the case which appear in the literature in the context of study branching random walks [1, 13]. Then we reduce the problem to the affine recursion in the critical case and applying theorem 1.6 one proves that $P[X > t] \sim C_+ t^{-1}$ (see [7] for more details).

2.2. **Inhomogeneous smoothing transform.** The inhomogeneous smoothing transform has been studied for a relatively short time. The problem of existence of solutions was investigated in recent papers of Alsmeyer and Meiners [2, 3]. Their results are similar to those described above (and also formulated in terms of the function m). They proved that if $m(\alpha) = 1$ and $m'(\alpha) < 0$ for some $\alpha \le 1$ (the contracting case) or $m(\alpha) = 1$ and $m'(\alpha) = 0$ for some $\alpha < 1$ (the critical case) then the set of solutions of (2.1) is not empty.

To study asymptotics one cannot reduce the problem as in the homogeneous case to the affine recursion. Nevertheless one can apply exactly the same methods, which give results for the affine recursion. This problem was studied by Jelenkovic and Olvera-Cravioto [14, 15] in the contracting case. Assuming that for some $\beta > \alpha$: $m(\beta) = 1$ and $m'(\beta) > 0$ and extending the Goldie's implicit renewal theory [11], they proved that $\mathbb{P}[X > t] \sim C_+ t^{-\beta}$. Positivity of the limiting constant C_+ was recently proved in [8]. The critical case is the subject of the forthcoming paper [9].

3. Proof of Theorem 1.6

3.1. **Preliminary estimates.** In order to prove that the sequence $\delta_{z^{-1}} * \nu$ has a limit, one has to prove first that divided by an appropriately chosen slowly varying function it is weakly convergent.

Proposition 3.1. *Suppose that* (1.4) *is satisfied and* $\log A$ *is aperiodic. Let* ν *be an invariant Radon measure not reduced to a mass point at* 0*. Then there exists a positive slowly varying function* L *on* $\mathbb{R}^+$ *such that the family of measures* $\frac{\delta_{z^{-1}} * \nu}{L(z)}$ *converges weakly to* $C\frac{da}{a}$ *for some strictly positive constant* C*.*

Proof. This proposition was indeed proved in [5] (Theorem 2.1). However the result stated there was written in the multidimensional settings and for this reason was slightly weaker than we need here. More precisely, it was proved in [5] that the family of measures is weakly compact and all accumulation points are invariant under the action of the group generated by the support of A. Nevertheless notice that in our settings this group is just $\mathbb{R}^+$, thus any accumulation point η must be of the form $\eta(da) = C_\eta \frac{da}{a}$. Moreover the slowly varying function is of the form $L(z) = \delta_{z^{-1}} * \nu(\Phi)$, where Φ a compactly supported Lipschitz function. Since

$$\lim_{z\to\infty} \frac{\delta_{z^{-1}} * \nu(\Phi)}{L(z)} = 1 = \eta(\Phi),$$

the constant C_η must be equal $\left(\int \Phi(a)\frac{da}{a}\right)^{-1}$ and does not depend on η. ☐

3.2. **Logarithmic estimates.** Proposition 3.1 implies in particular that the function $z \mapsto \nu(0, z)$ is bounded by some slowly varying function. Now we are going to prove that thanks to our addition assumptions this function is bounded just by a multiple of the logarithm.

For this purpose, let us recall the following [4] explicit construction of the measure ν. Define a random walk on $\mathbb{R}$

$$\begin{aligned} &S_0 = 0, \\ &S_n = \log(A_1 \dots A_n), \quad n \ge 1, \end{aligned} \tag{3.2}$$

and consider the downward ladder times of S_n:

$$\begin{aligned} &L_0 = 0, \\ &L_n = \inf\left\{k > L_{n-1}; S_k < S_{L_{n-1}}\right\}. \end{aligned} \tag{3.3}$$

Let $L = L_1$. The Markov process $\{X^x_{L_n}\}$ satisfies the recursion

$$X^x_{L_n} = M_n X^x_{L_{n-1}} + Q_n,$$

where (Q_n, M_n) is a sequence of i.i.d. random variables. Notice that $\{X_{L_n}\}$ is a contracting affine recursion possessing a stationary measure. Indeed since $\mathbb{E}[\log^2 A] < \infty$, we have $-\infty < \mathbb{E}S_L < 0$. Moreover $\mathbb{E}[\log^+(Q_n)] < \infty$ (see [21]). Therefore there exists a unique stationary measure ν_L of the process $\{X_{L_n}\}$. Define

$$\nu_0(f) = \int_{\mathbb{R}^+} \mathbb{E}\Big[\sum_{n=0}^{L-1} f(X^x_n)\Big]\nu_L(dx). \tag{3.4}$$

Then one can easily prove that ν_0 is μ invariant. At this point we cannot deduce that $\nu_0 = C\nu$ for some positive constant C, since we don't know whether ν_0 is a Radon measure. However this will be proved below.

Proposition 3.5. *Assume that* (1.4) *and* (1.8) *are satisfied. Then ν_0 is a multiple of ν. Moreover there exists a constant C such that for every bounded nonincreasing nonnegative function f on $\mathbb{R}^+$*

$$\int_{\mathbb{R}^+} f(u)\nu(du) < C\bigg(\|f\|_\infty + \int_{1/e}^\infty f(y)\frac{dy}{y}\bigg)$$

In particular for every $\varepsilon > 0$

$$\int_{\mathbb{R}^+} \frac{1}{\log^{1+\varepsilon}(2+u)}\nu(du) < \infty \tag{3.6}$$

and for $z > 1/e$

$$\nu(0,z) < C(2 + \log z). \tag{3.7}$$

Proof. Notice that since $X^x_n \ge A_1 \dots A_n x$

$$\nu(f) = \int_{\mathbb{R}^+} \mathbb{E}\bigg[\sum_{n=0}^{L-1} f(X^x_n)\bigg]\nu_L(dx) \le \int_{\mathbb{R}^+} \mathbb{E}\bigg[\sum_{n=0}^{L-1} f\big(e^{S_n}x\big)\bigg]\nu_L(dx).$$

Define the stopping time $T' = \inf\big\{n : \ S_n \ge 0\big\}$, where $S_n = \sum_{k=1}^n \log A_i$. Let $\{W_i\}$ be a sequence of i.i.d. random variables with the same distribution as the random variable $S_{T'}$ (recall $0 < \mathbb{E}S_{T'} < \infty$). Using the duality lemma we obtain

$$\nu_0(f) \le \int_{\mathbb{R}^+} \mathbb{E}\bigg[\sum_{n=0}^{L-1} f\big(e^{S_n}x\big)\bigg]\nu_L(dx) = \int_{\mathbb{R}^+} \mathbb{E}\bigg[\sum_{n=0}^{\infty} f\big(e^{W_1+\dots+W_n}x\big)\bigg]\nu_L(dx). \tag{3.8}$$

Let U be the potential associated with the random walk $W_1 + \ldots + W_n$, i.e.

$$U(a,b) = \mathbb{E}\big[\#n : \ a < W_1 + \ldots + W_n \le b\big].$$

By the renewal theorem $U(k, k+1)$ is bounded, thus we have

$$\begin{aligned} \nu_0(f) &\le \int_{\mathbb{R}^+} \mathbb{E}\bigg[\sum_{n=0}^{\infty} f\big(e^{W_1+\dots+W_n}x\big)\bigg]\nu_L(dx) \\ &\le \sum_{k=0}^{\infty}\int_{\mathbb{R}^+} U(k,k+1) f\big(e^k x\big)\nu_L(dx) \\ &\le C\sum_{k=0}^{\infty}\int_{\mathbb{R}^+} f\big(e^k x\big)\nu_L(dx). \end{aligned}$$

Next we divide the integral into two parts. First we estimate the integral over $(1,\infty)$

$$\begin{aligned}\sum_{k=0}^{\infty}\int_1^{\infty} f\big(e^k x\big)\nu_L(dx) &\le \sum_{k=0}^{\infty} f\big(e^k\big) \le \sum_{k=-1}^{\infty}\int_k^{k+1} f\big(e^y\big)dy\\ &= \int_{-1}^{\infty} f\big(e^y\big)dy = \int_{1/e}^{\infty} f(y)\frac{dy}{y}.\end{aligned}$$

Secondly, for $0<x<1$ we write

$$\begin{aligned}\sum_{k=0}^{\infty}\int_0^1 f\big(e^k x\big)\nu_L(dx) &\le \int_0^1\bigg(\sum_{k=0}^{|\log x|}+\sum_{k=|\log x|}^{\infty}\bigg) f\big(e^k x\big)\nu_L(dx)\\ &\le C\|f\|_\infty\int_0^1\big|\log x\big|\nu_L(dx)+\sum_{k=0}^{\infty} f\big(e^k\big)\\ &\le C\|f\|_\infty\int_0^1\big|\log x\big|\nu_L(dx)+\int_{1/e}^{\infty} f(y)\frac{dy}{y}.\end{aligned}$$

We will justify that the first term above is finite. Notice that if $x,y\in\mathbb{R}^+$ and $x+y<1$ then $\big|\log(x+y)\big|<\big|\log x\big|$. Observe also that $X^x_{L_n}\le X^y_{L_n}$ for $x\le y$. We write

$$\begin{aligned}\int_0^1\big|\log x\big|\nu_L(dx) &= \int_{\mathbb{R}^+}\mathbb{E}\Big[\big|\log X^x_L\big|\cdot\mathbf{1}_{\{X^x_L<1\}}\Big]\nu_L(dx)\\ &\le \int_{\mathbb{R}^+}\mathbb{E}\Big[\big|\log X^0_L\big|\cdot\mathbf{1}_{\{X^0_L<1\}}\Big]\nu_L(dx)\\ &\le \mathbb{E}\bigg[\Big|\log\Big(\frac{A_1A_2\dots A_LB_1}{A_1}\Big)\Big|\bigg]\\ &\le \mathbb{E}\big[|S_L|+|\log B_1|+|\log A_1|\big]<\infty.\end{aligned}$$

Therefore

$$\nu_0(f)\le C\bigg(\|f\|_\infty+\int_{1/e}^{\infty} f(y)\frac{dy}{y}\bigg).$$

Taking $f=\mathbf{1}_{[0,x]}$ we prove that ν_0 is a Radon measure, so ν is just a multiple of ν_0. In particular the last inequality is valid for ν instead of ν_0. Putting $f(u)=\frac{1}{\log^{1+\varepsilon}(2+u)}$ and next $f(u)=\mathbf{1}_{[0,z]}(u)$ we complete the proof. $\square$

3.3. **Translation of the invariant measure ν.** It will be convenient for our purpose to change slightly the measure ν and to consider the measure $\widetilde{\nu}$ defined by

$$\widetilde{\nu}(f)=\int_{\mathbb{R}^+} f(x-1)\nu(dx).$$

The crucial property of $\widetilde{\nu}$ is that its support is contained in $(1,\infty)$, so it does not contain 0, that allows us to avoid some technical problems. Let $\widetilde{\mu}$ be the law of the random pair $(A,A+B-1)$, then $\widetilde{\nu}$ is $\widetilde{\mu}$ invariant:

$$\begin{aligned}\widetilde{\mu}*\widetilde{\nu}(f) &= \mathbb{E}\bigg[\int_{\mathbb{R}^+} f\big(A(x+1)+B-1\big)\widetilde{\nu}(dx)\bigg]\\ &= \mathbb{E}\bigg[\int_{\mathbb{R}^+} f\big(Ax+B-1\big)\nu(dx)\bigg]=\int_{\mathbb{R}^+} f(x-1)\nu(dx)=\widetilde{\nu}(f).\end{aligned}$$

Notice that both measures ν and $\widetilde{\nu}$ have the same behavior at infinite, and the family of measure $\delta_{z^{-1}}*\nu$ and $\delta_{z^{-1}}*\widetilde{\nu}$, if converge do the same limit (of course assuming that they really converge, what we still have to prove). Thus, for our purpose it is sufficient to consider $\widetilde{\nu}$. However notice that although both measures μ and $\widetilde{\mu}$ are similar they satisfy slightly different hypotheses. The projections on the A-part of μ and $\widetilde{\mu}$ coincide and one can easily prove that $\widetilde{\mu}$ fulfills hypotheses (1.4) and (1.7). But the random variable $A+B-1$ may happen to be negative with positive probability, thus $\widetilde{\mu}$ may not satisfy assumption (1.8). Nevertheless, we are only interested in behaviour of ν and $\widetilde{\nu}$ at infinity, so we will use the fact, that we already know, that $\widetilde{\nu}$ satisfies both (3.6) and (3.7).

From now we consider measures $\widetilde{\nu}$ and $\widetilde{\mu}$ instead ν and μ, but to simplify our notation we will just write ν and μ. However the reader should be aware that we are in a slightly different settings and from now instead of (1.8) we assume:

- hypothesis (1.4) and (1.7) are satisfied;
- the measure ν satisfy (3.6) and (3.7).

3.4. **The Poisson equation.** In order to understand the asymptotic behavior of the measure ν one has to consider of the function

$$f_\phi(x) = \int_{\mathbb{R}^d} \phi(ue^{-x})\nu(du)$$

that is a solution of the Poisson equation

$$\overline{\mu} *_{\mathbb{R}} f_\phi = f_\phi + \psi_\phi \tag{3.9}$$

for a peculiar choice of the function ψ_ϕ, that is

$$\psi_\phi = \overline{\mu} *_{\mathbb{R}} f_\phi - f_\phi.$$

Under a number of assumptions concerning ψ_ϕ one can describe asymptotic behavior f_ϕ. Here we formulate the known results, based on the methods introduced by Port and Stone [23, 24], which we are going to use. For proofs we refer to [23, 5].

Let $\overline{\mu}$ be a centered aperiodic probability measure on $\mathbb{R}$ with the second moment $\sigma^2 = \int_{\mathbb{R}} x^2\overline{\mu}(dx)$. The Fourier transform of $\overline{\mu}$, $\widehat{\overline{\mu}}(\theta) = \int_{\mathbb{R}} e^{ix\theta}\overline{\mu}(dx)$ is a continuous bounded function, whose Taylor expansion near zero is $\widehat{\overline{\mu}}(\theta) = 1 + O(\theta^2)$ and such that $|1-\widehat{\overline{\mu}}(\theta)| > 0$ for all $\theta \in \mathbb{R}$. We consider the set $\mathcal{F}(\overline{\mu})$ of functions ψ that can be written as $\psi(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\theta}\widehat{\psi}(\theta)d\theta$ for some bounded, integrable, complex valued function $\widehat{\psi}$ verifying the following hypothesis

- its Taylor expansion near 0 is
$$\widehat{\psi}(\theta) = J(\psi) + i\theta K(\psi) + O(\theta^2)$$
for two constants $J(\psi)$ and $K(\psi)$,
- the function $\theta \mapsto \frac{\widehat{\psi}(-\theta)}{1-\widehat{\overline{\mu}}(\theta)} \cdot \mathbf{1}_{[-a,a]^c}(\theta)$ is integrable for some $a \in \mathbb{R}$.

The following result was proved in [5]

Theorem 3.10. *There exists a potential A, that is well defined on $\mathcal{F}(\overline{\mu})$ and such that $A\psi(x)$ is a continuous solution of the Poisson equation* (3.9)*. Furthermore if $J(\psi) \geq 0$ then $A\psi$ is bounded from below and*

$$\lim_{x\to\pm\infty} \frac{A\psi(x)}{x} = \pm\sigma^{-2}J(\psi). \tag{3.11}$$

If additionally $J(\psi) = 0$, then $A\psi$ is bounded and has a limit at infinity

$$\lim_{x\to\pm\infty} A\psi(x) = \mp\sigma^{-2}K(\psi). \tag{3.12}$$

Corollary 3.13. *If $J(\psi) = 0$, then every continuous solution of the Poisson equation bounded from below is of the form*

$$f = A\psi + C_0$$

for some constant C_0. Thus every continuous solution of the Poisson equation is bounded and the limit of $f(x)$ exists when x goes to $+\infty$.

Conversely if there exists a bounded solution of the Poisson equation, then $A\psi$ is bounded and $J(\psi) = 0$. In particular the first part of corollary is valid.

The next lemma describes a class of functions in $\mathcal{F}(\overline{\mu})$ that we will be used later on and that have the same type of decay at infinity as $\overline{\mu}$. In particular we see that if $\overline{\mu}$ has exponential moment then $\mathcal{F}(\overline{\mu})$ contains functions with exponential decay.

Lemma 3.14. *Let Y a random variable with the law $\overline{\mu}$, then the function*

$$r(x) = \mathbb{E}\left[|Y - x| - |x|\right]$$

is nonnegative and

$$\widehat{r}(\theta) = C \cdot \frac{\widehat{\overline{\mu}}(\theta) - 1}{\theta^2}$$

for $\theta \neq 0$. Moreover if $\mathbb{E}|Y|^{4+\varepsilon} < \infty$ for some $\varepsilon > 0$ then

$$r(x) \leq \frac{C}{1 + |x|^{3+\varepsilon}},$$

r is in $\mathcal{F}(\overline{\mu})$ and for every function $\zeta \in L^1(\mathbb{R})$ such that $x^2\zeta$ is integrable the convolution $r *_{\mathbb{R}} \zeta$ is in $\mathcal{F}(\overline{\mu})$.

Proof. The first part of the Lemma follows from the formula

$$r(x) = \begin{cases} -2\mathbb{E}[(Y + x)\mathbf{1}_{Y+x\leq 0}] & \text{for} \quad x \geq 0 \\ 2\mathbb{E}[(Y + x)\mathbf{1}_{Y+x>0}] & \text{for} \quad x < 0 \end{cases} \tag{3.15}$$

and was proved in [5]. For the second part we just notice, that the last formula implies for positive x:

$$\begin{aligned} |r(x)| &= 2\int_{y<-x} |y + x|\overline{\mu}(dy) = 2 \cdot \sum_{m=1}^{\infty} \int_{-(m+1)x\leq y<-mx} |y + x|\overline{\mu}(dy) \\ &\leq 2 \cdot \sum_{m=1}^{\infty} mx \int_{|y|>mx} \overline{\mu}(dy) \leq 2 \cdot \sum_{m=1}^{\infty} mx \int_{\mathbb{R}} \frac{|y|^{\chi}}{m^{4+\varepsilon}x^{4+\varepsilon}}\overline{\mu}(dy) \leq \frac{C}{x^{3+\varepsilon}}. \end{aligned}$$

It is clear that if $\mathbb{E}|Y|^{4+\varepsilon} < \infty$ then $r \in \mathcal{F}(\overline{\mu})$. If $\psi = r * \zeta$ with ζ and $x^2\zeta$ in $L^1(\mathbb{R})$ then it is easily checked that both ψ and $x^2\psi$ are integrable. Since $\widehat{\psi} = \widehat{r}\widehat{\zeta} = C\frac{\widehat{\overline{\mu}}-1}{\theta^2}\widehat{\zeta}$ and $\widehat{\zeta}$ vanish at infinity then $\psi \in \mathcal{F}(\overline{\mu})$ $\square$

Lemma 3.16. *If ϕ is a continuous function on $\mathbb{R}^+$ such that for $\beta > 2$*

$$|\phi(u)| \leq \frac{C}{(1 + \log^+ u)^{\beta}},$$

then the functions f_ϕ and $\overline{\mu} * f_\phi$ are well defined. Furthermore if ϕ is Lipschitz and $\beta > 4$, then

$$\int_{\mathbb{R}}\int_G\int_{\mathbb{R}^+} \Big|\phi(e^{-x}(au + b)) - \phi(e^{-x}au)\Big|\nu(du)\mu(db\, da)dx < \infty. \tag{3.17}$$

and

$$|\psi_\phi(x)| \leq \frac{C}{1 + |x|^{\chi}},$$

for $\chi = \min\{\beta - 1, 3 + \varepsilon\}$.

Proof. Assume first $x<-1$. In view of (3.7) we have

$$\begin{aligned}
\big|f_\phi(x)\big| &= \int_{u>1}\big|\phi(e^{-x}u)\big|\nu(du)\le \int_{u>1}\frac{C}{\log^\beta(e^{-x}u)}\nu(du)\\
&\le C\sum_{n=0}^\infty\int_{e^n\le u<e^{n+1}}\frac{1}{(n-x)^\beta}\nu(du)\\
&\le C\sum_{n>|x|}^\infty\frac{1}{n^\beta}\int_{e^{n+x}\le u<e^{n+x+1}}\nu(du)\\
&\le C\sum_{m=1}^\infty\sum_{m|x|\le n<(m+1)|x|}\frac{1}{m^\beta|x|^\beta}\int_{e^{n+x}\le u<e^{n+x+1}}\nu(du)\\
&\le C\sum_{m=1}^\infty\frac{1}{m^\beta|x|^\beta}\int_{u<e^{(m+1)|x|}}\nu(du)\le\frac{C}{|x|^{\beta-1}}\sum_{m=1}^\infty\frac{1}{m^{\beta-1}}\\
&\le\frac{C}{|x|^{\beta-1}}.
\end{aligned}$$

To proceed with positive x notice that, by (3.7), for every $y\in\mathbb{R}^+$ and $\beta'>2$, arguing as above, we obtain:

$$\begin{aligned}
\int_{\mathbb{R}^d}\frac{1}{1+\big(\log^+(y|u|)\big)^{\beta'}}\nu(du) &\le \int_{y|u|<1}\nu(du)+\sum_{n=0}^\infty\int_{e^n\le y|u|<e^{n+1}}\frac{1}{1+n^{\beta'}}\nu(du)\\
&\le C+C|\log y|+C\sum_{n=1}^\infty\frac{1}{1+n^{\beta'-1}}\le C(1+|\log y|)
\end{aligned}\tag{3.18}$$

Hence $|f_\phi(x)|\le C(1+x)$ if $x>0$.

Finally f_ϕ is continuous, hence for $x\in(-1,0)$ is bounded. Thus

$$|f_\phi(x)|\le C\left((1+|x|)\mathbf{1}_{x>0}+\frac{1}{1+|x|^{\beta-1}}\mathbf{1}_{x\le 0}\right)$$

Consider now the convolution of f_ϕ with $\overline{\mu}$. First if $x>0$, then

$$\big|\overline{\mu}*f_\phi(x)\big|\le C\int_{\mathbb{R}}\big(1+|x+y|\big)\overline{\mu}(dy)\le C\big(1+|x|\big).$$

Next if $x<-1$, then since $\mathbb{E}|\log A|^{4+\varepsilon}<\infty$, we have

$$\begin{aligned}
\big|\overline{\mu}*f_\phi(x)\big| &\le \int_{\mathbb{R}}\frac{C}{1+|x+y|^{\beta-1}}\overline{\mu}(dy)\\
&\le \int_{2|y|<|x|}\frac{C}{1+|x+y|^{\beta-1}}\overline{\mu}(dy)+\frac{C}{|x|^{4+\varepsilon}}\int_{2|y|\ge|x|}|y|^{4+\varepsilon}\overline{\mu}(dy)\\
&\le \frac{C}{1+|x|^{\chi_0}},
\end{aligned}$$

for $\chi_0=\min\{\beta-1,4+\varepsilon\}$. The function $\overline{\mu}*f_\phi$ is also continuous, hence finally we obtain

$$|\overline{\mu}*f_\phi(x)|\le C\left((1+|x|)\mathbf{1}_{x>0}+\frac{1}{1+|x|^{\chi_0}}\mathbf{1}_{x\le 0}\right).$$

Next we have

$$
\begin{aligned}
&\int_{-\infty}^{0}\int_G\int_{\mathbb{R}^d}\Big|\phi(e^{-x}(au+b))-\phi(e^{-x}au)\Big|\nu(du)\mu(db\,da)dx\\
&\quad\le \int_{-\infty}^{0}\int_G\int_{\mathbb{R}^d}\big|\phi(e^{-x}(au+b))\big|\nu(du)\mu(db\,da)dx+\int_{-\infty}^{0}\int_G\int_{\mathbb{R}^d}\big|\phi(e^{-x}au)\big|\nu(du)\mu(db\,da)dx\\
&\qquad\qquad\le \int_{-\infty}^{0}\int_{\mathbb{R}^d}\big|\phi(e^{-x}u)\big|\nu(du)dx+\int_{-\infty}^{0}\int_G\int_{\mathbb{R}^d}\big|\phi(e^{-x}au)\big|\nu(du)\mu(db\,da)dx\\
&\qquad\qquad\qquad\qquad\qquad\qquad\le \int_{-\infty}^{0}\big|f_{|\phi|}(x)\big|dx+\int_{-\infty}^{0}\big|\overline{\mu}*f_{|\phi|}(x)\big|dx
\end{aligned}
$$

and in view of our previous estimates both integrals above are finite.

For $x>0$ we divide the integral of $\big|\phi(e^{-x}au)-\phi(e^{-x}(b+au))\big|$ into several parts and we use the following inequality, being a consequence of the Lipschitz property of ϕ:

$$
|\phi(s)-\phi(r)|\le C|s-r|^{\theta}\max_{\xi\in\{|s|,|r|\}}\frac{1}{1+(\log^+\xi)^{\beta'}},
$$

where $\theta<1-2/\beta$ and $\beta'=\beta(1-\theta)>2$. We denote by μ_A the law of A.

Case 1. First we assume $|b|\le e^{\frac{x}{2}}$. Then by (3.18)

$$
\begin{aligned}
&\int_{|b|\le e^{\frac{x}{2}}}\int_{\mathbb{R}^+}\Big|\phi(e^{-x}au)-\phi(e^{-x}(b+au))\Big|\nu(du)\mu(db\,da)\\
&\le C\int_{|b|\le e^{\frac{x}{2}}}\int_{\mathbb{R}^+}e^{-\theta x}|b|^{\theta}\left(\frac{1}{1+(\log^+(e^{-x}a|u|))^{\beta'}}+\frac{1}{1+(\log^+(e^{-x}|au+b|))^{\beta'}}\right)\nu(du)\mu(db\,da)\\
&\le Ce^{-\theta x/2}\left(\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{1}{1+(\log^+(e^{-x}a|u|))^{\beta'}}\nu(du)\mu_A(da)+\int_{\mathbb{R}^+}\frac{1}{1+(\log^+(e^{-x}|u|))^{\beta'}}\nu(du)\right)\\
&\qquad\qquad\qquad\qquad\le Ce^{-\theta x/2}\left[1+x+\int_{\mathbb{R}^+}|\log a|\mu_A(da)\right]<Ce^{-\theta x/4}.
\end{aligned}
$$

Case 2. We assume $au<2|au+b|$ and $|b|>e^{\frac{x}{2}}$. Notice first

$$
\int_{|b|>e^{\frac{x}{2}}}\mu(db\,da)\le\frac{C}{1+x^{4+\varepsilon}}\int_{\mathbb{R}^+}\Big(1+\big(\log^+|b|\big)^{4+\varepsilon}\Big)\mu(db\,da)\le\frac{C}{1+x^{4+\varepsilon}}.
$$

and

$$
\begin{aligned}
&\int_{|b|>e^{\frac{x}{2}}}(|\log a|+\log|b|)\,\mu(db\,da)\\
&\qquad\le\frac{C}{1+x^{3+\varepsilon}}\int_G(1+\big(|\log a|+\log^+|b|\big)^{3+\varepsilon}(\log^+|b|+|\log a|)\mu(db\,da)\le\frac{C}{1+x^{3+\varepsilon}}.
\end{aligned}
$$

Then, proceeding as previously, we have

$$\begin{aligned}
&\int\int_{\substack{a|u|<2|au+b|\\ |b|>e^{\frac{x}{2}}}}\Big|\phi(e^{-x}au)-\phi(e^{-x}(b+au))\Big|\nu(du)\mu(db\,da)\\
&\qquad\le 2\int\int_{\substack{a|u|<2|au+b|\\ |b|>e^{\frac{x}{2}}}}\max\Big\{\big|\phi(e^{-x}au)\big|,\big|\phi(e^{-x}(b+au))\big|\Big\}\nu(du)\mu(db\,da)\\
&\qquad\qquad\le C\int_{|b|>e^{\frac{x}{2}}}\int_{\mathbb{R}^d}\frac{1}{1+(\log^+(e^{-x}a|u|))^\beta}\nu(du)\mu(db\,da)\\
&\qquad\qquad\qquad\le C\int_{|b|>e^{\frac{x}{2}}}\big(x+|\log a|+1\big)\mu(da\,db)\le\frac{C}{1+x^{3+\varepsilon}}.
\end{aligned}$$

Case 3. The last case is $a|u|\ge 2|au+b|$ and $|b|>e^{\frac{x}{2}}$. Then $|u|<\frac{2|b|}{a}$ and we obtain

$$\begin{aligned}
&\int\int_{\substack{a|u|\ge2|au+b|\\ |b|>e^{\frac{x}{2}}}}\Big|\phi(e^{-x}au)-\phi(e^{-x}(b+au))\Big|\nu(du)\mu(db\,da)\\
&\le C\int_{|b|>e^{\frac{x}{2}}}\int_{|u|<\frac{2|b|}{a}}\nu(du)\mu(db\,da)\le C\int_{|b|>e^{\frac{x}{2}}}\big(1+\log|b|+|\log a|\big)\mu(db\,da)\le\frac{C}{1+x^{3+\varepsilon}}.
\end{aligned}$$

We conclude (3.17) and the required estimates for ψ_ϕ. □

Proof of Theorem 1.6. First, we are going to prove that the limit

$$\lim_{x\to+\infty}\int_{\mathbb{R}^+}\phi(ue^{-x})\nu(du)=T(\phi):=-2\sigma^{-2}K(\psi_\phi) \tag{3.19}$$

exists and is finite for a class of very particular functions, namely for functions of the form

$$\phi(u)=\int_{\mathbb{R}}r(t)\zeta(e^tu)dt, \tag{3.20}$$

where

$$r(t)=\mathbb{E}\left[|-\log A_1-t|-|t|\right] \tag{3.21}$$

and ζ is a nonnegative Lipschitz function on $\mathbb{R}^+$ such that $\zeta(u)\le e^{-\gamma|\log|u||}$ for some $\gamma>0$.

For this purpose we are going to prove prove that ψ_ϕ is an element of $\mathcal{F}(\overline{\mu})$ and $J(\psi_\phi)=0$. Then, by Corollary 3.13, the function $f_\phi(x)$, is a solution of the corresponding Poisson equation, and thus it is bounded and has a limit when x converge to $+\infty$.

In view of (1.8),

$$\begin{aligned}
|\phi(u)| &\le C\int_{\mathbb{R}}\frac{1}{1+|t-\log|u||^{3+\varepsilon}}e^{-\gamma|t|}dt\\
&\le \frac{C}{1+|\log|u||^{3+\varepsilon}}\int_{\mathbb{R}}\frac{1+|t-\log|u||^{3+\varepsilon}+|t|^{3+\varepsilon}}{1+|t-\log|u||^{3+\varepsilon}}e^{-\gamma|t|}dt\\
&\le \frac{C}{1+|\log|u||^{3+\varepsilon}}\int_{\mathbb{R}}(1+|t|^{3+\varepsilon})e^{-\gamma|t|}dt\le\frac{C}{1+|\log|u||^{3+\varepsilon}}.
\end{aligned}$$

Thus by Lemma 3.16, f_ϕ, f_ζ, $\overline{\mu}*f_\phi$ and $\overline{\mu}*f_\zeta$ are well defined. Furthermore since ζ is Lipschitz ψ_ζ is bounded, and $x^2\psi_\zeta(x)$ is integrable on $\mathbb{R}$. We cannot guarantee that ϕ is Lipschitz, but we can observe that

$$f_\phi(x)=\int_{\mathbb{R}^d}\int_{\mathbb{R}}r(t)\zeta(e^{-x+t}u)dt\nu(du)=\int_{\mathbb{R}^d}\int_{\mathbb{R}}r(t+x)\zeta(e^tu)dt\,\nu(du)=r*_{\mathbb{R}}f_\zeta(x)$$

and

$$\overline{\mu} * f_\phi(x) = r *_{\mathbb{R}} (\overline{\mu} * f_\zeta)(x).$$

Hence

$$\psi_\phi = f_\phi - \overline{\mu} * f_\phi = r * (f_\zeta - \overline{\mu} * f_\zeta) = r *_{\mathbb{R}} \psi_\zeta.$$

Therefore, by Lemma 3.14, $\psi_\phi \in \mathcal{F}(\overline{\mu})$.

Furthermore $J(\psi_\phi) = 0$. In fact,

$$\begin{aligned} \int_{\mathbb{R}} \psi_\zeta(x)dx &= \int_G \int_{\mathbb{R}^+} \int_{\mathbb{R}} \left[\zeta\Big(e^{-x+\log(|au|)}\Big) - \zeta_r\Big(e^{-x+\log|au+b|}\Big) \right] dx\nu(du)\mu(db\,da) \\ &= \int_G \int_{\mathbb{R}^+} \left(\int_{\mathbb{R}} \zeta(e^{-x})dx - \int_{\mathbb{R}} \zeta(e^{-x})dx \right) \nu(du)\mu(db\,da) = 0. \end{aligned}$$

Observe that we can apply the Fubini theorem since ζ is Lipschitz and, by Lemma 3.16, the absolute value of the integrand in the second line above is integrable. Hence

$$J(\psi_\phi) = \int_{\mathbb{R}} \psi_\phi(x)dx = \int_{\mathbb{R}} r * \psi_\zeta(x)dx = \int_{\mathbb{R}} r(x)dx \cdot \int_{\mathbb{R}} \psi_\zeta(x)dx = 0.$$

By Corollary 3.13, we have

$$f_\phi = A\psi_\phi + C_\phi \tag{3.22}$$

where C_ϕ is a constant. Thus, f_ϕ is bounded.

In particular the same holds for f_{Φ_γ}, where

$$\Phi_\gamma(u) = \int_{\mathbb{R}} r(t)e^{-\gamma|t+\log|u||}dt.$$

Since zero does not belong to the support of ν, $\lim_{x\to-\infty} f_\phi(x) = 0$ and by Theorem 3.10

$$-C_\phi = \lim_{x\to-\infty} A\psi_\phi(x) = \sigma^{-2}K(\psi_\phi).$$

Thus when x goes to $-\infty$ the limit of h_ϕ exists which is possible only if h_ϕ is constant and is equal to $-\sigma^{-2}K(\psi_\phi)$. Finally

$$\lim_{x\to+\infty} f_\phi(x) = \lim_{x\to+\infty} A\psi_\phi(x) - \sigma^{-2}K(\psi_\phi) = -2\sigma^{-2}K(\psi_\phi)$$

and we obtain (3.19).

Fix a $\gamma > 0$. Since $\Phi_\gamma > 0$ for every function $\phi \in C_c(\mathbb{R}^+)$ there exists a constant C_ϕ such that $|\phi| \le C_\phi\Phi_\gamma$. Thus the family of measures on $\mathbb{R}^+$

$$\delta_{(0,e^{-x})} *_G \nu(\phi) = \int_{\mathbb{R}^+} \phi(e^{-x}u)\nu(du)$$

is bounded, hence it is relatively compact in the weak topology. Let η be an accumulation point for a subsequence $\{x_n\}$ that is

$$\lim_{n\to\infty} \delta_{(0,e^{-x_n})} *_G \nu(\phi) = \eta(\phi) \quad \forall\phi \in C_c(\mathbb{R}^+). \tag{3.23}$$

The measure η is $\mathbb{R}^+$ invariant [5], thus η must be of the form $\eta(da) = C_\eta \frac{da}{a}$. A standard argument proves indeed that for any continuous non negative function such that $\phi \le C_\phi\Phi_\gamma$, not necessarily compactly supported,

$$\eta(\phi) = \lim_{n\to\infty} \delta_{(0,e^{-x_n})} *_G \nu(\phi).$$

In particular the last formula holds for $\Phi_\gamma(u) = \int_{\mathbb{R}} r(t)e^{-\gamma|t+\log|u||}dt$, since $\eta(\Phi_\gamma) = C_\eta \int_{\mathbb{R}^*_+} \Phi_\gamma(a)\frac{da}{a}$. Then:

$$C_\eta = \frac{T(\Phi_\gamma)}{\int_{\mathbb{R}^*_+} \Phi_\gamma(a)\frac{da}{a}}$$

does not depend on η. Thus, finally, we deduce that the limit

$$\lim_{z\to+\infty} \int_{\mathbb{R}^+} \phi(uz^{-1})\nu(du)$$

exists for every function $\phi \in C_c(\mathbb{R}^+)$ and defines a Radon measure Λ on $\mathbb{R}^+$. This limiting measure must be $\mathbb{R}^+$ invariant, therefore is of the form $C\frac{da}{a}$, that by a standard argument implies also (1.9). To prove that C is strictly positive we proceed as in [5].

□

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