

POSITIVE HARMONIC FUNCTIONS FOR SEMI-ISOTROPIC RANDOM WALKS ON TREES, LAMPLIGHTER GROUPS, AND DL-GRAPHS

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ABSTRACT. We determine all positive harmonic functions for a large class of “semi-isotropic” random walks on the lamplighter group, i.e., the wreath product $\mathbb{Z}_q \wr \mathbb{Z}$, where $q \geq 2$. This is possible via the geometric realization of a Cayley graph of that group as the Diestel-Leader graph $DL(q, q)$. More generally, $DL(q, r)$ ($q, r \geq 2$) is the horocyclic product of two homogeneous trees with respective degrees $q + 1$ and $r + 1$, and our result applies to all DL-graphs. This is based on a careful study of the minimal harmonic functions for semi-isotropic walks on trees.

1. INTRODUCTION

Let X be an infinite, connected, locally finite graph X with root vertex o , and P the transition matrix $P = (p(x, y))_{x, y \in X}$ of a random walk $(Z_n)_{n \geq 0}$ on X . That is, $Z_n \in X$ is the random position of the random walker at time n , and $\Pr[Z_{n+1} = y \mid Z_n = x] = p(x, y)$. The n -step transition probability $p^{(n)}(x, y) = \Pr[Z_n = y \mid Z_0 = x]$, $x, y \in X$, is the (x, y) -entry of the matrix power P^n , with $P^0 = I$, the identity matrix. The *Green kernel* is

$$(1.1) \quad G(x, y) = \sum_{n=0}^{\infty} p^{(n)}(x, y).$$

We suppose here that P is *irreducible*: $G(x, y) > 0$ for all $x, y \in X$, and *transient*: $G(x, y) < \infty$.

A function $h : X \rightarrow \mathbb{R}$ is called *harmonic*, or *P-harmonic*, if $h = Ph$, where $Ph(x) = \sum_y p(x, y) h(y)$, and *superharmonic* if $h \geq Ph$. A function $h \in \mathcal{H}^+$, the cone of positive harmonic functions, is called *minimal* if $h(o) = 1$, and $h \geq h_1 \in \mathcal{H}^+$ implies that h_1/h is constant. The minimal harmonic functions are the extreme points of the convex base $\mathcal{B} = \{h \in \mathcal{H}^+ : h(o) = 1\}$ of the cone \mathcal{H}^+ . Every positive harmonic function has a unique integral representation with respect to a Borel measure on the set of minimal ones, see DOOB [10].

Positive harmonic functions for various classes of random walks (Markov chains) have been a continuous subject of study since the 1950ies. One of the typical questions is to determine and describe all positive harmonic functions in terms of a geometric or algebraic

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structure of the underlying graph X , to which the transition probabilities are assumed to be adapted. See the monograph by WOESS [27], Ch. IV, for various results in this spirit.

In the present paper, we determine all minimal, and thereby also all positive harmonic functions for a large class of random walks on the *lamplighter group* $\mathbb{Z}_q \wr \mathbb{Z}$. This is the *wreath product* of the additive group $\mathbb{Z}_q = \{0, \dots, q-1\}$ of integers modulo $q (\geq 2)$ with the group \mathbb{Z} of all integers.

The lamplighter interpretation is as follows: \mathbb{Z} represents an infinite street, i.e. the graph with edges $[k, k+1]$, $k \in \mathbb{Z}$, with a lamp at the midpoint $k - \frac{1}{2}$ of each edge. Each lamp can have q different states in \mathbb{Z}_q ; the state 0 corresponds to the lamp being switched off. A lamplighter wanders along \mathbb{Z} (from a point k to $k \pm 1$), and at each step, he may change the state of the lamp on the edge which he traverses. (Below, we will also allow bigger “jumps” along \mathbb{Z} and changing the lamps on more than one of the nearby edges). For a corresponding random walk, the information that we have to keep track of at each instant is the pair (η, k) , where k is the current position of the lamplighter, and η is the current configuration of the states of the lamps.

We remark that for the usual construction of the wreath product, one thinks of the lamps sitting at the points of \mathbb{Z} . For our purpose, it is more convenient to have them (equivalently) sitting at the edges’ midpoints, i.e., the elements of $\mathbb{Z} - \frac{1}{2}$. In these terms, the formal construction of the lamplighter group is as follows. Consider the group of all finitely supported *configurations* $\mathcal{C} = \{\eta : \mathbb{Z} - \frac{1}{2} \rightarrow \mathbb{Z}_q, \text{supp}(\eta) \text{ finite}\}$ with pointwise addition modulo q . Then \mathbb{Z} acts on \mathcal{C} by translations $k \mapsto T_k : \mathcal{C} \rightarrow \mathcal{C}$ with $T_k \eta(m - \frac{1}{2}) = \eta(m - k - \frac{1}{2})$. The resulting semidirect product $\mathbb{Z} \ltimes \mathcal{C}$ is

$$\mathbb{Z}_q \wr \mathbb{Z} = \{(\eta, k) : \eta \in \mathcal{C}, k \in \mathbb{Z}\}, \quad \text{group operation} \quad (\eta, k)(\eta', k') = (\eta + T_k \eta', k + k').$$

The group identity is $o = (\mathbf{0}, 0)$, where $\mathbf{0}$ is the zero configuration. For two pairs $x = (\eta, k)$, $y = (\eta', k')$, we define the left and right *flags*

$$(1.2) \quad \begin{aligned} f_1 &= f_1(x, y) = \min\{k, k', m : \eta(m - \frac{1}{2}) \neq \eta'(m - \frac{1}{2})\} \quad \text{and} \\ f_2 &= f_2(x, y) = \max\{k, k', m : \eta(m + \frac{1}{2}) \neq \eta'(m + \frac{1}{2})\}. \end{aligned}$$

These are the left- and rightmost positions on \mathbb{Z} which the lamplighter is forced to visit if he starts at k with configuration η and wants to reach k' with configuration η' , when at each single step he traverses a single edge and is allowed to change the state of the lamp on that edge. We also define the corresponding increments

$$(1.3) \quad u_1 = k - f_1 \quad \text{and} \quad u_2 = f_2 - k.$$

We say that a random walk on $\mathbb{Z}_q \wr \mathbb{Z}$ is *semi-isotropic* if the transition probability from $x = (\eta, k)$ to $y = (\eta', k')$ depends only on u_1 , u_2 and $k - k'$. Every random walk of this type is adapted to the group structure of $\mathbb{Z}_q \wr \mathbb{Z}$. Indeed, $p(x, y) = \mu(x^{-1}y)$, where $\mu(x) = p(o, x)$, a probability measure on $\mathbb{Z}_q \wr \mathbb{Z}$.

A typical class of examples can be obtained as follows. For $m \in \mathbb{Z} \setminus \{0\}$, let μ_m be the probability measure associated with the random walk, where from position k and configuration η , the lamplighter jumps to $k + m$ and switches each of the lamps on the $|m-1|$ edges between k and $k + m$ to a uniformly chosen random state (independently of each other), while leaving the other lamps unchanged. Write $\mu_0 = \delta_o$, the point mass

at the identity. If $\tilde{\mu}$ is any probability measure on \mathbb{Z} then the probability measure

$$(1.4) \quad \mu = \sum_m \tilde{\mu}(m) \mu_m$$

gives rise to a semi-isotropic random walk. The latter is irreducible if and only if the random walk on \mathbb{Z} induced by $\tilde{\mu}$ is irreducible. Also, it is transient, since $\mathbb{Z}_q \wr \mathbb{Z}$ has exponential growth, see VAROPOULOS [25] or the exposition in [27], Ch. I.

There are natural projections $\pi_1, \pi_2 : \mathbb{Z}_q \wr \mathbb{Z} \rightarrow \mathbb{T}_q$, where \mathbb{T}_q is the homogenous tree with degree $q + 1$. Under each of the two projections, every semi-isotropic transition matrix P projects to transition matrices P_1 and P_2 (respectively) on \mathbb{T}_q , which are also semi-isotropic in an adequate sense.

Our results arise as a special case of a more general class of lamplighter type random walks, which – as well as the ones discussed so far – arise as random walks on the *Diestel-Leader graphs* $DL(q, r)$, where $q, r \geq 2$. In that description, the projections π_1, π_2 map $DL(q, r)$ onto \mathbb{T}_q and \mathbb{T}_r , respectively. The details will be explained in §2.

Our first result, Theorem 3.4, states that every minimal P -harmonic function h is of the form $h(x) = h_1(\pi_1 x)$ or $h(x) = h_2(\pi_2 x)$, where h_i is a P_i -harmonic function on \mathbb{T}_q ($i = 1, 2$). This leads us to a careful study of all minimal harmonic functions for semi-isotropic random walks on \mathbb{T}_q . This is done under suitable moment conditions in Theorem 4.23, on the basis of recent work of BROFFERIO [4]. In Theorem 5.1, we then describe all minimal, and thereby all positive P -harmonic functions on $DL(q, r)$. The results and their proofs are a considerable extension as well as simplification of those of WOESS [28], who only dealt with the nearest neighbour case. We also remark here that another extension of [28] is given by BROFFERIO AND WOESS [5], who study only nearest neighbour random walks, but give precise asymptotic estimates (in space) of the Green kernel, which leads to a description of the full *Martin compactification*. The latter contains more analytical information than the one provided by knowledge of the minimal harmonic functions. However, it seems hard to extend the methods of [5] to general semi-isotropic random walks as considered in the present paper.

In concluding the introduction, let us remark that the first to show that lamplighter groups are fascinating objects in the study of random walks were KAIMANOVICH AND VERSHIK [18]. By now, there is a considerable amount of literature on this topic, regarding various issues. See e.g. KAIMANOVICH [17], LYONS, PEMANTLE AND PERES [19], ERSCHLER [13], [14], REVELLE [23], [24], BERTACCHI [3], GRIGORCHUK AND ŽUK [15], DICKS AND SCHICK [8], BARTHOLDI AND WOESS [2], SALOFF-COSTE AND PITTET [21], [22].

We also remark here that in part, our methods have their roots in the study of random walks on the affine group over the real, resp. p -adic numbers, see ELIE [12], BABILLOT, BOUGEROL AND ELIE [1], resp. CARTWRIGHT, KAIMANOVICH AND WOESS [7] and BROFFERIO [4].

2. LAMPLIGHTERS AND DIESTEL-LEADER GRAPHS

We now explain very briefly the structure of the DL-graphs and their relation with the wreath products $\mathbb{Z}_q \wr \mathbb{Z}$. See also [28], [2] and [5]; here we choose a different order of

explanations, which together with those of the latter papers may create a more complete picture.

Consider the two-way-infinite path \mathbb{Z} with lamps sitting at the midpoints of the edges, as above. However, we now think of a more general model, where each lamp may be switched on in two different colours, green and red. There are q possible green states (intensities), encoded by \mathbb{Z}_q , and r possible red states, encoded by \mathbb{Z}_r . In both cases, the respective 0 state means “switched off”. Only finitely many lamps may be switched on, and the rule is that all lamps on the left (towards $-\infty$) of the lamplighter have to be in a green state, while all lamps on the right (towards $+\infty$) have to be in a red state. For each $k \in \mathbb{Z}$, let

$$\mathcal{C}_k = \left\{ \eta : \mathbb{Z} - \frac{1}{2} \rightarrow \mathbb{Z}_q \dot{\cup} \mathbb{Z}_r \mid \text{supp}(\eta) \text{ finite}, \eta(m - \frac{1}{2}) \in \mathbb{Z}_q \forall m \leq k, \eta(m + \frac{1}{2}) \in \mathbb{Z}_r \forall m \geq k \right\}.$$

The state space of our lamplighter walks, i.e., the vertex set of the DL-graph, is the set $X = \bigcup_{k \in \mathbb{Z}} \mathcal{C}_k \times \{k\}$. Two pairs (η, k) and (η', k') are neighbours if $k' = k \pm 1$ and η coincides with η' everywhere except at the midpoint of the edge between k and k' . This describes the Diestel-Leader graph $\text{DL}(q, r)$.

In general, we consider \mathbb{Z}_q and \mathbb{Z}_r as being disjoint, but when $q = r$, we need not distinguish between the two colours. In the latter case, we can omit the index k of \mathcal{C}_k . It is obvious from this description that $\text{DL}(q, q)$ is a Cayley graph of $\mathbb{Z}_q \wr \mathbb{Z}$, since the latter group acts on $\text{DL}(q, q)$ transitively and fixed-point-freely by graph isometries.

We next want to achieve a geometric understanding of the graph structure. If $\eta \in \mathcal{C}_k$ then set

$$\eta_k^-(m) = \eta(k - m - \frac{1}{2}) \quad \text{and} \quad \eta_k^+(m) = \eta(k + m + \frac{1}{2}), \quad m \geq 0.$$

That is, we split η at k and consider the left (η_k^-) and right (η_k^+) halves as elements of Σ_q and Σ_r , respectively, where

$$\Sigma_q = \left\{ (\sigma(m))_{m \geq 0} : \sigma(\cdot) \in \mathbb{Z}_q, \text{supp}(\sigma) \text{ finite} \right\}.$$

(In [28], these sequences were indexed over the non-positive integers.) The set $\Sigma_q \times \mathbb{Z}$ is in one-to-one correspondence with the homogeneous tree \mathbb{T}_q of degree $q + 1$. Seen as a vertex of that tree, each element (σ, k) has a unique *predecessor* $(\sigma', k - 1)$, where $\sigma'(m) = \sigma(m + 1)$ for $m \geq 0$, that is, σ' is obtained by deleting $\sigma(0)$ from σ . This describes neighbourhood, so that (σ, k) has precisely q *successors* - the elements that have (σ, k) as their predecessor.

We can draw this tree in horocyclic layers. The k -th *horocycle* is $H_k = \Sigma_q \times \{k\}$, and for every element of H_k , its predecessor lies on H_{k-1} , while its successors lie on H_{k+1} . See Figure 1a below, and Figure 1 in [28] for a more detailed picture. The closure (as a partial order) of the predecessor relation is the *ancestor* relation. Every pair of vertices $x_1 = (\sigma, k)$ and $y_1 = (\sigma', k')$ has an infimum, i.e., a maximal common ancestor. Formally, this is $x_1 \wedge y_1 = (\bar{\sigma}, \bar{k})$, where

$$\bar{k} = \max \left\{ \ell \leq \min\{k, k'\} : \sigma|_{[k-\ell, \infty)} \equiv \sigma'|_{[k'-\ell, \infty)} \right\} \quad \text{and} \quad \bar{\sigma}(m) = \sigma(k - \bar{k} + m), \quad m \geq 0.$$

In this notation, setting $\mathbf{u}(x_1, y_1) = k - \bar{k}$ (whence $\mathbf{u}(y_1, x_1) = k' - \bar{k}$), the graph distance in \mathbb{T}_q is $d(x_1, y_1) = \mathbf{u}(x_1, y_1) + \mathbf{u}(y_1, x_1)$, and $\mathbf{u}(y_1, x_1) = d(x_1, x_1 \wedge y_1)$. Also, $\mathbf{u}(y_1, x_1) - \mathbf{u}(x_1, y_1) = \mathfrak{h}(y_1) - \mathfrak{h}(x_1)$, where $\mathfrak{h}(x_1) = k$ if $x \in H_k$. See Figure 1b.

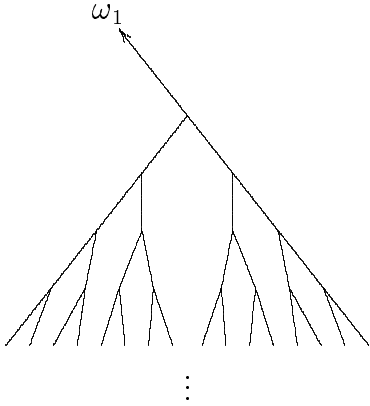


Figure 1a

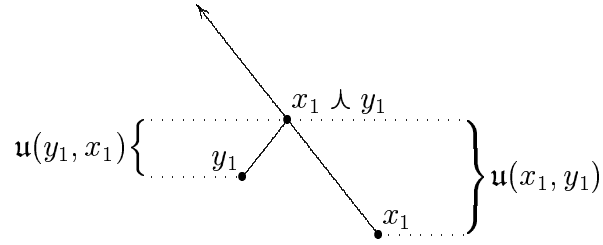


Figure 1b

Given $\text{DL} = \text{DL}(q, r)$, we write $\mathbb{T}^1 = \mathbb{T}_q$ and $\mathbb{T}^2 = \mathbb{T}_r$ and define the projections $\pi_i : \text{DL} \rightarrow \mathbb{T}_i$ as follows. If $x = (\eta, k)$ then $\pi_1 x = x_1 := (\eta_k^-, k)$ and $\pi_2 x = x_2 := (\eta_k^+, -k)$. Each of these mappings is a neighbourhood-preserving surjection of DL onto the respective tree. Conversely, starting with the trees,

$$\text{DL}(q, r) = \{x = x_1 x_2 : x_i \in \mathbb{T}^i, \mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0\}$$

with neighbourhood given by $x_1 x_2 \sim y_1 y_2 \iff x_1 \sim y_1$ and $x_2 \sim y_2$. Again, a detailed geometric picture can be found in [28], Figure 2. In that reference (as well as in [2] and [5]), the explanation follows the reversed order, starting with the geometric description. We recall that when $x = x_1 x_2, y = y_1 y_2 \in \text{DL}$ then

$$(2.1) \quad \mathfrak{u}(x_1, y_1) + \mathfrak{u}(x_2, y_2) = \mathfrak{u}(y_1, x_1) + \mathfrak{u}(y_2, x_2),$$

and their distance is, by [3],

$$(2.2) \quad d(x, y) = d(x_1, y_1) + d(x_2, y_2) - |\mathfrak{h}(x_1) - \mathfrak{h}(y_1)|.$$

With the 2-colour-lamplighter interpretation, the left and right flags can be defined as in (1.2), and if $x = (\eta, k), y = (\eta', k')$ then

$$(2.3) \quad \mathfrak{f}_i = \mathfrak{h}(x_i \wedge y_i), \quad i = 1, 2.$$

Every DL -graph is vertex-transitive (its isometry group acts transitively on the vertex set), but only when $r = q$ it is a Cayley graph of a finitely generated group.

3. POSITIVE HARMONIC FUNCTIONS ON DL -GRAPHS

A random walk, resp. its transition matrix P_i on \mathbb{T}^i ($i = 1, 2$) is called *semi-isotropic*, if $p_i(x_i, y_i)$ depends only on $\mathfrak{u}(x_i, y_i)$ and $\mathfrak{u}(y_i, x_i)$, or equivalently, only on $d(y_i, x_i)$ and $\mathfrak{h}(y_i) - \mathfrak{h}(x_i)$. (Recall that a random walk is called isotropic if $p_i(x_i, y_i)$ depends only on the distance.)

Analogously, we call a random walk, resp. its transition matrix P on DL *semi-isotropic*, if $p(x, y)$ depends only on the four numbers $\mathfrak{u}(x_1, y_1)$, $\mathfrak{u}(y_1, x_1)$, $\mathfrak{u}(x_2, y_2)$ and $\mathfrak{u}(y_2, x_2)$,

which must satisfy (2.1). That is, there is a probability measure \mathbf{m} on the set $\{(k_1, l_1, k_2, l_2) \in \mathbb{N}_0^4 : k_1 + k_2 = l_1 + l_2\}$ (where \mathbb{N}_0 denotes the non-negative integers) such that

$$(3.1) \quad p(x, y) = \mathbf{m}(\mathbf{u}(x_1, y_1), \mathbf{u}(y_1, x_1), \mathbf{u}(x_2, y_2), \mathbf{u}(y_2, x_2))$$

for all $x = x_1x_2, y = y_1y_2 \in \text{DL}(q, r)$. In terms of the lamplighter, the transition probability from (η, k) to (η', k') depends only on the distances of k and k' to the left and right flags (1.2), (2.3).

The projection P_1 of P on \mathbb{T}^1 is given by

$$p_1(x_1, y_1) = \sum_{y_2: y_1y_2 \in \text{DL}} p(x_1x_2, y_1y_2),$$

which is independent of the specific choice of $x_2 \in \mathbb{T}^2$ such that $x_1x_2 \in \text{DL}$. The projection P_2 on \mathbb{T}^2 is analogous. Both are semi-isotropic along with P .

(3.2) Lemma. *Every semi-isotropic, irreducible random P walk on $\text{DL}(q, r)$ is transient, and its projections P_i on \mathbb{T}^i ($i = 1, 2$) are also transient.*

Proof. This follows from Theorem 5.13 in [27], since $\text{DL}(q, r)$ as well as the \mathbb{T}^i are vertex-transitive graphs with exponential growth. \square

Let f be a function defined on $\text{DL}(q, r)$. When we say that f depends only on x_1 , resp. f depends only on x_2 , then this means

$$f(x_1x_2) = f(x_1y_2) \quad \forall x_1x_2, x_1y_2 \in \text{DL}, \quad \text{resp.} \quad f(x_1x_2) = f(y_1x_2) \quad \forall x_1x_2, y_1x_2 \in \text{DL}.$$

In the first case, we can write $f(x_1x_2) = f_1(x_1)$, where f_1 is a function on \mathbb{T}^1 , and in the second case, $f(x_1x_2) = f_2(x_2)$, where f_2 is a function on \mathbb{T}^2 . Note that when both conditions hold, then this does not mean that f is constant, but that $f(x_1x_2) = \tilde{f}(\mathfrak{h}(x_1))$, where \tilde{f} is a function on \mathbb{Z} . The following is an obvious exercise.

(3.3) Lemma. *A function h_1 is P_1 -harmonic on \mathbb{T}^1 if and only if $h(x_1x_2) = h_1(x_1)$ is P -harmonic on DL .*

Here is the first main main result, along with a surprisingly simple proof.

(3.4) Theorem. *Suppose that P is semi-isotropic and irreducible on DL . Then the following statements hold.*

(a) *Every minimal P -harmonic function on $\text{DL}(q, r)$ is of the form $h(x_1x_2) = h_1(x_1)$ or $h(x_1x_2) = h_2(x_2)$, where h_i is a minimal P_i -harmonic function on \mathbb{T}^i , $i = 1, 2$ (respectively).*

(b) *If h is a positive P -harmonic function on $\text{DL}(q, r)$, then there are non-negative P_i -harmonic functions h_i on \mathbb{T}^i ($i = 1, 2$) such that*

$$h(x_1x_2) = h_1(x_1) + h_2(x_2) \quad \forall x_1x_2 \in \text{DL}(q, r).$$

We shall appeal to *Martin boundary* theory for Markov chains, see DOOB [10], HUNT [16], or the excellent introduction by DYNKIN [11]. If $G(\cdot, \cdot)$ is the Green kernel of any

transient, irreducible Markov chain with countable state space X , then the *Martin kernel* is

$$K(x, y) = G(x, y)/G(o, y), \quad x, y \in X,$$

where $o \in X$ is a reference point. The *Martin compactification* is the smallest compact Hausdorff space \widehat{X} containing X as a dense, discrete subset, such that for each $x \in X$, the function $K(x, \cdot)$ extends continuously to \widehat{X} . The extended kernel on $X \times \widehat{X}$ is also denoted $K(\cdot, \cdot)$. The *Martin boundary* is $\mathcal{M} = \widehat{X} \setminus X$. A basic result of the theory is that every minimal harmonic function is of the form $K(\cdot, \zeta)$, where $\zeta \in \mathcal{M}$, and that the *minimal Martin boundary* \mathcal{M}_{\min} , consisting of all $\zeta \in \mathcal{M}$ for which $K(\cdot, \zeta)$ is minimal harmonic, is a Borel subset of \mathcal{M} . The *Poisson-Martin representation theorem* says that for every positive harmonic function h there is a unique Borel measure ν^h on \mathcal{M} such that

$$(3.5) \quad \nu^h(\mathcal{M} \setminus \mathcal{M}_{\min}) = 0 \quad \text{and} \quad h(x) = \int_{\mathcal{M}} K(x, \cdot) d\nu^h \quad \forall x \in X.$$

Proof of Theorem 3.4. Recall that our “root” (origin) of DL is $o = (\mathbf{0}, 0)$. Let $h = K(\cdot, \zeta)$ be a minimal harmonic function. Then there is a sequence $y^{(n)} = y_1^{(n)} y_2^{(n)} \in \text{DL}$ such that

$$K(x, y^{(n)}) \rightarrow h(x) \quad \forall x \in \text{DL}.$$

Then, by (2.1) and (2.2), at least one of the sequences $(\mathbf{u}(o_i, y_i^{(n)}))_n$ is unbounded. (Recall that for any $x \in \text{DL}$, we write $x_i = \pi_i x$ for its projection on the tree \mathbb{T}^i .) Thus, passing to a subsequence, we assume that $\mathbf{u}(o_1, y_1^{(n)}) \rightarrow \infty$ (first case). We claim that in this case, $h(x)$ depends only on x_2 . Indeed, fix $x = x_1 x_2 \in \mathbb{T}^2$ and let v_1 lie on the same horocycle of \mathbb{T}^1 as x_1 , so that also $v = v_1 x_2 \in \text{DL}$. Let $k = \mathbf{u}(x_1, v_1) = \mathbf{u}(v_1, x_1)$. By assumption, $\mathbf{u}(x_1, y_1^{(n)}) \rightarrow \infty$ as $n \rightarrow \infty$. Therefore there is $n(k)$ such that $\mathbf{u}(x_1, y_1^{(n)}) > k$ for all $n \geq n(k)$. This implies

$$(3.6) \quad \mathbf{u}(x_1, y_1^{(n)}) = \mathbf{u}(v_1, y_1^{(n)}) \quad \text{and} \quad \mathbf{u}(y_1^{(n)}, x_1) = \mathbf{u}(y_1^{(n)}, v_1) \quad \forall n \geq n(k),$$

see Figure 2.

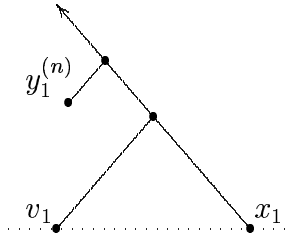


Figure 2

Formula (3.1) implies that also the Green kernel is semi-isotropic, and consequently

$$K(x, y^{(n)}) = \frac{G(x_1 x_2, y_1^{(n)} y_2^{(n)})}{G(o_1 o_2, y_1^{(n)} y_2^{(n)})} = \frac{G(v_1 x_2, y_1^{(n)} y_2^{(n)})}{G(o_1 o_2, y_1^{(n)} y_2^{(n)})} = K(v, y^{(n)}).$$

for all $n \geq n(k)$. Letting $n \rightarrow \infty$, we obtain $h(x) = h(v)$. Lemma 3.3 yields that $h(x) = h_2(x_2)$, where h_2 is a P_2 -harmonic function on \mathbb{T}^2 . Minimality of h as a P -harmonic function implies minimality of h_2 as a P_2 -harmonic function. (The converse is in general not true.)

By exchanging the roles of the two trees, we see that if $u(o_2, y_2^{(n)}) \rightarrow \infty$ (second case) then $h(\cdot)$ depends only on x_2 , and $h(x) = h_1(x_1)$, where h_1 is a minimal P_1 -harmonic function on \mathbb{T}^1 .

This proves (a). To see (b), let

$$\mathcal{M}_i(P) = \{\zeta \in \mathcal{M}(P) : K(\cdot, \zeta) \text{ is minimal and depends only on } x_i\}, \quad i = 1, 2.$$

Then $\mathcal{M}_{\min}(P) = \mathcal{M}_1(P) \cup \mathcal{M}_2(P)$. (The two pieces are not necessarily disjoint, see below.) The topology of the Martin boundary is the one of pointwise convergence of the Martin kernels, and \mathcal{M}_{\min} is a Borel subset. Since the set of $\mathcal{S}_i(P)$ of all positive P -superharmonic functions that depend only on x_i is closed with respect to pointwise convergence¹, $\mathcal{M}_i(P) = \mathcal{M}_{\min}(P) \cap \mathcal{S}_i(P)$ is a Borel subset of the Martin boundary. Thus, if ν^h is as in (3.5), then $h = h_1 + h_2$, where

$$(3.7) \quad h_1 = \int_{\mathcal{M}_1(P)} K(\cdot, \zeta) d\nu^h(\zeta) \quad \text{and} \quad h_2 = \int_{\mathcal{M}_2(P) \setminus \mathcal{M}_1(P)} K(\cdot, \zeta) d\nu^h(\zeta).$$

By (a), h_i is P_i -harmonic for $i = 1, 2$. □

4. SEMI-ISOTROPIC RANDOM WALKS ON A HOMOGENEOUS TREE

In view of Theorem 3.4, our next aim is determine those minimal harmonic functions for P_i on \mathbb{T}^i ($i = 1, 2$) which lift to minimal P -harmonic functions on $\text{DL}(q, r)$. Note that when h_i is minimal harmonic for P_i on \mathbb{T}^i then $h(x_1 x_2) = h_i(x_i)$ is not necessarily minimal for P on DL (while the converse *is* true), compare with WOESS [28].

In the following we shall omit the subscripts i for elements of \mathbb{T}^i . Thus, *in the present section*, P denotes an irreducible, semi-isotropic random walk on $\mathbb{T} = \mathbb{T}_q$.

We recall the construction of the geometric boundary $\partial\mathbb{T}$. A *geodesic ray* is a one-sided infinite path in \mathbb{T} without repeated vertices. A boundary point (end) is an equivalence class of rays, where two rays are equivalent if they differ only by finitely many initial points. If $x \in \mathbb{T}$ and $w \in \widehat{\mathbb{T}} = \mathbb{T} \cup \partial\mathbb{T}$ then there is a unique geodesic path from x to w (if $w \in \mathbb{T}$), resp. ray representing w (if $w \in \partial\mathbb{T}$) starting at x , denoted by \overline{xw} . Analogously, if ξ, η are distinct ends, then there is a unique two-sided infinite geodesic path $\overline{\xi\eta}$ whose two ‘halves’ (when split at any of its vertices) represent ξ and η .

The *confluent* $c(v, w)$ of $v, w \in \widehat{\mathbb{T}}$ is the last common vertex on the geodesics from the origin $o \in \mathbb{T}$ to v and w , respectively. Writing $|x| = d(x, o)$ for $x \in \mathbb{T}$, we can equip $\widehat{\mathbb{T}}$ with the ultrametric $\theta(v, w) = q^{-|c(v, w)|}$, if $v \neq w$ (and $\theta(v, v) = 0$). Thus, $\widehat{\mathbb{T}}^1$ is a compact space with \mathbb{T} discrete, open and dense.

Let ω be the end of \mathbb{T} represented by the ray $[o = o_0, o_1, o_2, \dots]$ where each o_k is the predecessor of o_{k-1} , and the root o is on the horocycle H_0 . In the correspondence between vertices of \mathbb{T} and sequences described in §2, we have $o_k = (\mathbf{0}, -k)$. Then $\omega = \omega_1$ is the end of $\mathbb{T} = \mathbb{T}^1$ located at the top of the picture in Figure 1a, while the ends in $\partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\omega\}$ are located at the bottom of that picture. Each $\xi \in \partial\mathbb{T}$ corresponds to a two-sided infinite

¹When P does not have finite range, we cannot take positive P -harmonic functions here, since a pointwise limit of such functions is not necessarily harmonic, while being superharmonic by Fatou's theorem

sequence in $\mathbb{Z}_q^{\mathbb{Z}}$ for which there is $\bar{m} \in \mathbb{Z}$ such that $\xi(n) = 0$ for all $n \leq \bar{m}$. Given $x \in \mathbb{T}$ and $\xi \in \partial^*\mathbb{T}$, we can define $x \wedge \xi$ and $\mathbf{u}(x, \xi)$ in the same way as above (see Figure 1b).

Now consider the Martin kernel $K(x, y) = G(x, y)/K(o, y)$ associated with P on \mathbb{T} . In the case when P has *bounded range*, i.e., $p(x, y) = 0$ when $d(x, y) > R$ ($R < \infty$) then it is easy to find the minimal P -harmonic functions as a consequence of a general result of PICARDELLO AND WOESS [20] regarding transient, bounded range random walks on arbitrary trees.

(4.1) Proposition. *If P is irreducible and has bounded range then the associated Martin compactification is the end compactification $\widehat{\mathbb{T}}$, and each extended kernel $K(\cdot, \xi)$, $\xi \in \partial\mathbb{T}$ is a minimal harmonic function for P .*

We want to extend the resulting description of the minimal Martin boundary in two ways. First, we go beyond bounded range, and second, we shall describe the resulting minimal Martin kernels in more computational detail.

Invariant measures on the boundary. The transition probabilities of P are invariant under the locally compact, totally disconnected group $\Gamma = \text{Aff}(\mathbb{T})$ of all isometries of \mathbb{T} that fix ω . This group acts transitively on \mathbb{T} , so that we can interpret our random walk as a random walk on that group via the construction described, e.g., in WOESS [26], §3. Namely, normalize the left Haar measure dg on Γ such that the stabilizer Γ_o of o in Γ (which is an open-compact subgroup) has measure 1. Define a probability measure μ on Γ by

$$(4.2) \quad \mu(dg) = p(o, go) dg.$$

Let X_n , $n \geq 1$, be a sequence of i.i.d. Γ -valued random variables with common distribution μ . Consider the *right random walk*

$$R_n = X \cdots X_n$$

on Γ . Then, given $g \in \Gamma$, the sequence $gR_n o$ is (a model of) the random walk on \mathbb{T} with transition matrix P and starting point $x = go$. In particular, all results of CARTWRIGHT, KAIMANOVICH AND WOESS [7] and BROFFERIO [4] regarding random walks on Γ apply here. Since the action of Γ extends to $\partial^*\mathbb{T}$, we can convolve μ with any (Radon) measure ν on $\partial^*\mathbb{T}$. If $E \subset \partial^*\mathbb{T}$ is a Borel set, then

$$\mu * \nu(E) = \int_{\Gamma} \nu(g^{-1}E) \mu(dg).$$

We are looking for an *invariant* measure ν , satisfying $\mu * \nu = \nu$. It will serve to describe the minimal harmonic functions. For $x \in \mathbb{T}$, $\xi \in \partial^*\mathbb{T}$ and $k, r \geq 0$, let

$$(4.3) \quad \begin{aligned} T_{k,r}(x) &= \{y \in \mathbb{T} : \mathbf{u}(x, y) = k, \mathbf{u}(y, x) = r\}, \\ \Omega_k(x) &= \{\eta \in \partial^*\mathbb{T} : \mathbf{u}(x, \eta) = k\} \quad \text{and} \quad \Omega(\xi) = \Omega_0(x_0), \end{aligned}$$

where x_0 is the element with $\mathfrak{h}(x_0) = 0$ on the geodesic $\overline{\omega\xi}$. If $x = o$ then we just write Ω_k and $T_{k,r}$. The sets $\Omega_0(x)$, $x \in \mathbb{T}$, are open-compact and generate the topology of $\partial^*\mathbb{T}$.

(4.4) Lemma. *If μ is defined by (4.2) and ν is μ -invariant on $\partial^*\mathbb{T}$, then ν is equidistributed on each set Ω_k , that is, for each $l \geq 1$ and $y \in T_{k,r}$ with $r \geq 1$,*

$$\nu(\Omega_0(y)) = \nu(\Omega_k)/|T_{k,r}|.$$

Proof. By (4.2), the measure μ is invariant under the stabilizer of o : if $g \in \Gamma_o$ then $\delta_g * \mu = \mu$. Therefore also $\delta_g * \nu = \nu$. If $y, z \in T_{k,r}$ then there is $g \in \Gamma_o$ such that $gy = z$. Thus $\nu(\Omega_0(z)) = \delta_g * \nu(\Omega_0(z)) = \nu(g^{-1}\Omega_0(z)) = \nu(\Omega_0(y))$. \square

The following is now the result of a straightforward computation of the numbers $\mu * \nu(\Omega_j)$, $j \geq 0$.

(4.5) Lemma. *If μ and ν are as in Lemma 4.4, and*

$$\mu_{k,r} = \sum_{x \in T_{k,r}} p(o, x),$$

then the measure ν is determined by the values $a_j = \nu(\Omega_j)$, $j \geq 0$. The latter must satisfy the following equations.

$$a_0 = \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \frac{a_r}{(q-1)q^{k-1}} \mu_{k,r} + \sum_{k=1}^{\infty} \frac{a_0}{q^k} \mu_{k,0} + \sum_{r=0}^{\infty} (a_0 + \cdots + a_r) \mu_{0,r}$$

and for each $j \geq 1$,

$$\begin{aligned} a_j &= \sum_{k=0}^{j-1} \sum_{r=0}^{\infty} a_{j+r-k} \mu_{k,r} + \sum_{k=j+1}^{\infty} \sum_{r=1}^{\infty} \frac{a_r}{q^{k-j}} \mu_{k,r} \\ &\quad + \frac{q-1}{q} \sum_{k=j}^{\infty} \frac{a_0}{q^{k-j}} \mu_{k,0} + \sum_{r=1}^{\infty} \left(a_0 + \cdots + a_{r-1} + \frac{q-2}{q-1} a_r \right) \mu_{j,r}. \end{aligned}$$

In particular, if $p(x, y) = 0$ whenever $d(x, y) > N$ (bounded range), then

$$(4.6) \quad a_j = \sum_{n=-N}^N a_{j+n} \tilde{\mu}(n) \quad \text{for all } j > N, \quad \text{where } \tilde{\mu}(n) = \sum_{r-k=n} \mu_{k,r}.$$

Next, we look for sufficient conditions that guarantee the existence (and uniqueness) of a solution of the above system for the a_j , or equivalently, of a μ -invariant measure ν on $\partial^*\mathbb{T}$.

Our basic requirement is that P has *finite first moment* $m(P)$, where more generally the moment of order $t > 0$ is defined as

$$(4.7) \quad m_t(P) = \sum_x d(o, x)^t p(o, x)$$

(This is a generic definition, whenever we have a transition matrix and a metric.)

We can consider the projection \tilde{P} of P onto \mathbb{Z} , where for arbitrary $x \in \mathbb{T}$ with $\mathfrak{h}(x) = k$,

$$(4.8) \quad \tilde{p}(k, l) = \sum_{y: \mathfrak{h}(y)=l} p(x, y) = \tilde{\mu}(l - k),$$

with $\tilde{\mu}(n)$ as in (4.6). This defines an irreducible, translation-invariant random walk on \mathbb{Z} which also has finite first moment, so that we can define its *drift*

$$(4.9) \quad \alpha(P) = \alpha(\tilde{P}) = \sum_{n \in \mathbb{Z}} n \tilde{p}(0, n).$$

(4.10) Proposition. *Suppose that P is semi-isotropic and irreducible on $\mathbb{T} = \mathbb{T}_q$.*

(a) *If $m(P) < \infty$ and $\alpha(P) > 0$ then up to constant multiples, there is a unique μ -invariant measure ν on $\partial^* \mathbb{T}$, and its total mass $\nu(\partial^* \mathbb{T})$ is finite.*

(b) *If $m_{2+\varepsilon}(P) < \infty$ (where $\varepsilon > 0$) and $\alpha(P) = 0$ then up to constant multiples, there is a unique μ -invariant measure ν on $\partial^* \mathbb{T}$, and $\nu(\partial^* \mathbb{T}) = \infty$.*

In both cases, ν is supported by the whole of $\partial^ \mathbb{T}$.*

Proof. (a) In this case, it is well known [7] that the random walk $Z_n = R_n o$ starting at o converges in the topology of $\tilde{\mathbb{T}}$ to a $\partial^* \mathbb{T}$ -valued random variable Z_∞ , and ν is its distribution. Irreducibility of P implies that ν is supported by the whole boundary. Uniqueness follows from the fact that $R_n \xi \rightarrow Z_\infty$ almost surely for every $\xi \in \partial^* \mathbb{T}$, see [7].

(b) This is proved in [4], Prop. 2.4. \square

The case $\alpha(P) < 0$ is different. For general, semi-isotropic P , consider the function

$$(4.11) \quad \varphi(c) = \sum_{m \in \mathbb{Z}} \tilde{\mu}(m) e^{cm}.$$

It is well known that irreducibility implies that $\varphi : \mathbb{R} \rightarrow (0, \infty]$ is convex (strictly convex where it is finite) and that $\lim_{c \rightarrow \pm\infty} \varphi(c) = \infty$. Thus, there are at most two solutions to the equation $\varphi(c) = 1$. We have $\varphi(0) = 1$, and $\varphi'(0) = \alpha(P)$, if the derivative exists. In particular, if $\alpha(P) < 0$ and $\varphi(c_0) = 1$ for some $c_0 \neq 0$, then it must be $c_0 > 0$; a sufficient condition for the existence of such a value c_0 is that $\limsup_{n \rightarrow \infty} \tilde{\mu}(n)^{1/n} = 0$. Given c_0 , we define a new transition matrix P^\sharp and associated probability measure μ^\sharp on Γ by

$$(4.12) \quad p^\sharp(x, y) = p(x, y) e^{c_0(\mathfrak{h}(y) - \mathfrak{h}(x))} \quad \text{and} \quad \mu^\sharp(dg) = e^{c_0 \mathfrak{h}(g_0)} \mu(dg)$$

P is stochastic (since $\varphi(c_0) = 1$), irreducible, and inherits semi-isotropy from P .

(4.13) Proposition. *If there is $c_0 \in \mathbb{R}$ such that*

$$(4.14) \quad c_0 > 0, \quad \varphi(c_0) = 1, \quad \text{and} \quad \sum_{x_i} d(o_i, x_i) p(o_i, x_i) e^{c_0 \mathfrak{h}(x_i)} < \infty,$$

then for μ^\sharp as in (4.12), up to constant multiples, there is a unique μ^\sharp -invariant measure ν on $\partial^ \mathbb{T}$. Its support is $\partial^* \mathbb{T}$, and $\nu(\partial^* \mathbb{T}) < \infty$.*

Proof. First of all, note that here we did not require existence of the first moment. However, if $m(P) < \infty$, then (4.14) implies $\alpha(P) < 0$ because of the shape of the function $\varphi(c)$.

Consider P^\sharp on \mathbb{T} , defined in (4.12). Condition (4.14) says that $m(P^\sharp) < \infty$, whence $\alpha(P^\sharp)$ is finite. If $\alpha(P^\sharp)$ were non-negative, then we could not have $\varphi(c) = 1$ for any $c < c_0$, contradicting the fact that $\varphi(0) = 1$. Therefore, $\alpha(P^\sharp) > 0$, and we can apply

Proposition 4.10(a) to P^\sharp and the associated probability measure μ^\sharp on Γ . We find the unique μ^\sharp -invariant probability measure ν on $\partial^*\mathbb{T}$. \square

(4.15) Remark. In order to compute the coefficients a_j associated with ν via Lemma 4.5, one has to replace the numbers $\mu_{k,r}$ with $\mu_{k,r}^\sharp = \mu_{k,r} e^{c_0(r-k)}$.

Harmonic measures and Radon-Nikodym derivatives.

A. Non-negative drift. Suppose that $m(P) < \infty$ and $\alpha(P) > 0$, or that $m_{2+\varepsilon}(P) < \infty$ and $\alpha(P) = 0$. Let ν be the invariant measure (probability measure in the first case) according to Proposition 4.10. Define

$$\nu_x = \delta_g * \nu, \quad \text{where } x \in \mathbb{T} \text{ and } g \in \Gamma \text{ with } go = x,$$

that is, $\nu_x(E) = \nu(g^{-1}E)$ for Borel sets $E \subset \partial^*\mathbb{T}$. Since ν is Γ_o -invariant, the measure ν_x is independent of the specific choice of g with $go = x$. We observe that when $\alpha(P) > 0$ then $\nu_x(E)$ is the probability that the random walk governed by P and starting at x converges to a point in E . We have

$$(4.16) \quad \nu_x = \sum_y p(x, y) \nu_y \quad \text{for every } x \in \mathbb{T}.$$

Therefore, we call the family of measures $(\nu_x)_{x \in \mathbb{T}}$ *harmonic measures*.

B. Negative drift. Suppose that (4.14) holds, and let ν be the unique μ^\sharp -invariant probability measure on $\partial^*\mathbb{T}$ according to Proposition 4.13. This time, define

$$\nu_x = e^{c_0 b(x)} \delta_g * \nu, \quad \text{where } x \in \mathbb{T} \text{ and } g \in \Gamma \text{ with } go = x.$$

Then again, the measures $(\nu_x)_{x \in \mathbb{T}}$ satisfy (4.16).

(4.17) Proposition. *Under the moment conditions of (4.10) and (4.14), respectively, the harmonic measures are mutually absolutely continuous. For each $x \in \mathbb{T}$, the Radon-Nikodym derivative $d\nu_x/d\nu_o$ has a continuous realization, which we denote by $K(x, \cdot)$.*

If k, l are such that $\Omega_k(o) \cap \Omega_l(x) \neq \emptyset$, then

$$K(x, \xi) = \frac{\nu_x(\Omega_k(o) \cap \Omega_l(x))}{\nu_o(\Omega_k(o) \cap \Omega_l(x))} \quad \text{for all } \xi \in \Omega_k(o) \cap \Omega_l(x).$$

(Note that the nonempty ones among the sets $\Omega_k(o) \cap \Omega_l(x)$, $k, l \geq 0$, form a partition of $\partial^*\mathbb{T}$ consisting of open-compact parts.)

Proof. Let $x, y \in \mathbb{T}$. By irreducibility, there is n such that $p^{(n)}(x, y) > 0$. Since

$$\nu_x = \sum_w p^{(n)}(x, w) \nu_w \geq p^{(n)}(x, y) \nu_y,$$

we find that $\nu_y \ll \nu_x$. Since ν_x is equidistributed and does not vanish on any $\Omega_k(x)$, the above formula for the Radon-Nikodym derivative is immediate. As the sets $\Omega_k(o) \cap \Omega_l(x)$ are open, it also follows that $K(x, \cdot)$ is locally constant, whence continuous. \square

(4.16) implies that for each $\xi \in \partial^*\mathbb{T}$, the function $x \mapsto K(x, \xi)$ is harmonic, and $K(o, \xi) = 1$ by construction.

We can compute $K(x, \xi)$ more explicitly in terms of the coefficients a_j of Lemma 4.5, taking into account Remark 4.15 when the “negative drift” condition (4.14) holds. The following is obtained by a lengthy, but completely straightforward discussion of all possible relative positions of ξ , x and o .

(4.18) Lemma. *Let $x \in \mathbb{T}$ and $\xi \in \partial^*\mathbb{T}$. Set $k = \mathbf{u}(o, \xi)$, $l = \mathbf{u}(x, \xi)$ and $m = \mathbf{h}(x)$, so that $k, l \geq 0$ and $m \in \mathbb{Z}$. Also, set $b(m) = 1$ in case A (non-negative drift) and $b(m) = e^{com}$ in case B (negative drift). Then*

$$K(x, \xi) = b(m) \frac{a_l}{a_k} q^{k-l+m} \left(\frac{q}{q-1} \right)^{\epsilon(k,l,m)},$$

where

$$\begin{aligned} \epsilon(k, l, m) &= \text{sign}(k - l + m) \text{sign}(k) \text{sign}(l), \quad \text{except for} \\ \epsilon(0, l, m) &= 1, \quad 1 \leq l \leq m \quad (m > 0) \quad \text{and} \quad \epsilon(k, 0, m) = -1, \quad 1 \leq k \leq -m \quad (m < 0). \end{aligned}$$

In particular, the P -harmonic function $K(\cdot, \xi)$ is unbounded for every $\xi \in \partial^*\mathbb{T}$.

Our notation seems to indicate that each function $K(\cdot, \xi)$ is a Martin kernel. This is indeed true.

(4.19) Theorem. *Suppose that P is irreducible, and that the respective moment conditions of cases A or B above hold. Let $G(x, y)$ and $K(x, y) = G(x, y)/G(o, y)$ be the associated Green and Martin kernels. Then, with $K(x, \xi)$ as defined in Proposition 4.17, we have*

$$\lim_{n \rightarrow \infty} K(x, y_n) = K(x, \xi),$$

whenever (y_n) is a sequence of vertices that tends to $\xi \in \partial^*\mathbb{T}$ in the topology of the end compactification $\widehat{\mathbb{T}}$.

Proof. We only need to consider case A. Indeed, in case B, when (4.14) holds, then $m(P^\sharp) < \infty$ and $\alpha(P^\sharp) > 0$. Since $K(x, y) = e^{c\mathbf{h}(x)} K^\sharp(x, y)$, the result will follow from case A.

The proof is based on BROFFERIO [4], Thm. 3.6.2–3. We briefly explain how that result has to be “translated” to our situation. Let μ be as in (4.2), and let ν be the μ -invariant measure on $\partial^*\mathbb{T}$ (unique up to normalization) according to Proposition 4.10.

Besides the group $\Gamma = \text{Aff}(\mathbb{T})$ and the stabilizer Γ_o , we also need the *horocyclic subgroup* $\text{Hor}(\Gamma) = \{g \in \Gamma : \mathbf{h}(go) = 0\}$ of all elements of Γ that stabilize some (whence every) horocycle as a set. Let $\mathbf{K} = \mathbf{K}_\xi$ denote the stabilizer of $\xi \in \partial^*\mathbb{T}$ in $\text{Hor}(\Gamma)$. It is a compact subgroup, and since $\text{Hor}(\Gamma)$ acts transitively on $\partial^*\mathbb{T}$, we can identify $\partial^*\mathbb{T}$ with $\text{Hor}(\Gamma)/\mathbf{K}$. Thus, we can lift ν to a right- \mathbf{K} -invariant measure $\bar{\nu}$ on $\text{Hor}(\Gamma)$. More precisely, for each $\eta \in \partial^*\mathbb{T}$, let g_η be an element of $\text{Hor}(\Gamma)$ with $g_\eta \xi = \eta$. Also, let $\lambda_{\mathbf{K}}$ be the Haar measure of \mathbf{K} , normalized with total mass 1. Then, for any Borel set $B \subset \text{Hor}(\Gamma)$,

$$\bar{\nu}(B) = \int_{\partial^*\mathbb{T}} \lambda_{\mathbf{K}}(g_\eta^{-1} \{g \in B : g\xi = \eta\}) d\nu(\eta).$$

This is independent of the specific choice of g_η , but $\bar{\nu}$ does depend on ξ . In particular, if the set B is right- \mathbf{K} -invariant, then $\bar{\nu}(B) = \nu(B\xi)$.

Next, given $\xi \in \partial^*\mathbb{T}$, we can choose a “shift” $s = s_\xi$ in Γ that maps each element of the geodesic $\overline{\omega\xi}$ to its successor on $\overline{\omega\xi}$. Let $\lambda_{\langle s \rangle}$ be the counting measure on the cyclic subgroup $\langle s \rangle$ of Γ .

We can consider both $\bar{\nu}$ and $\lambda_{\langle s \rangle}$ as measures on the whole of Γ , supported by the respective subgroups. We also consider the potential associated with μ , that is, the Radon measure $\mathcal{U} = \sum_{n=0}^{\infty} \mu^{(n)}$, where $\mu^{(n)}$ is the n -th convolution power of μ . By transience, \mathcal{U} is a Radon measure on Γ . Then Thm. 3.6 of [4] says that in case A, with the proper choice of the normalizing constant $c(\mu) > 0$,

$$(4.20) \quad \mathcal{U} * \delta_{g_n^{-1}} \rightarrow c(\mu) \cdot \bar{\nu} * \lambda_{\langle s \rangle} \quad \text{vaguely, as } n \rightarrow \infty,$$

whenever (g_n) is a sequence in Γ such that $g_n o \rightarrow \xi$ in the end topology. (As a matter of fact, [4] states and proves the “inverse” statement, and we are applying that result to the reflected measure $\check{\mu}$.)

We now translate this to our situation. Recall that the stabilizer Γ_o is open and compact, and also recall the definition (4.2) of μ in terms of left Haar measure on Γ . It implies that $\mathcal{U}(g\Gamma_o) = \Delta(g^{-1})\mathcal{U}(\Gamma_o g)$ for every $g \in \Gamma$, where Δ is the modular function of Γ . It is known that for $g \in \Gamma = \text{Aff}(\mathbb{T}_q)$, the latter is $\Delta(g) = q^{\mathfrak{h}(g o)}$, see e.g. [7]. For y_n tending to ξ , we find $g_n \in \Gamma$ such that $g_n x = y_n$. We obtain for the Green kernel of P

$$G(o, y_n) = \mathcal{U}(g_n \Gamma_o) = q^{-\mathfrak{h}(y_n)} \mathcal{U} * \delta_{g_n^{-1}}(\Gamma_o).$$

Consequently, (4.20) implies

$$q^{\mathfrak{h}(y_n)} G(o, y_n) \rightarrow c(\mu) \cdot \bar{\nu} * \lambda_{\langle s \rangle}(\Gamma_o) = c(\mu) \cdot \bar{\nu}(\Gamma_o)$$

In order to compute $\bar{\nu}(\Gamma_o)$, observe that $\Gamma_o \mathbf{K}$ is the disjoint union of $|\mathbf{K}o|$ right Γ_o -cosets, and that $|\mathbf{K}o| = |T_{k,k}|$, where $T_{k,k}$ is as in (4.3) and $k = \mathbf{u}(o, \xi)$. Since the measure $\bar{\nu}$ and the set $\Gamma_o \mathbf{K}$ are right- \mathbf{K} -invariant,

$$|T_{k,k}| \bar{\nu}(\Gamma_o) = \bar{\nu}(\Gamma_o \mathbf{K}) = \nu(\Gamma_o \mathbf{K} \xi) = \nu(\Gamma_o \xi) = \nu(\Omega_k).$$

We obtain that $\bar{\nu}(\Gamma_o) = \nu(\Omega(\xi))$, as defined in (4.3).

Now we observe that for any $g \in \Gamma$ and $y \in \mathbb{T}$, setting $\eta = g^{-1}\xi$, we have

$$\mathfrak{h}(g^{-1}y) = \mathfrak{h}(y) - \mathfrak{h}(g o) \quad \text{and} \quad \nu(\Omega(\eta)) = q^{-\mathfrak{h}(g o)} \delta_g * \nu(\Omega(\xi)).$$

Therefore, for any $g \in G$, setting $\eta = g^{-1}\xi$,

$$\begin{aligned} q^{\mathfrak{h}(y_n) - \mathfrak{h}(g o)} G(g o, y_n) &= q^{\mathfrak{h}(g^{-1}y_n)} G(o, g^{-1}y_n) \\ &\rightarrow c(\mu) \cdot \nu(\Omega(\eta)) = c(\mu) \cdot q^{-\mathfrak{h}(g o)} \delta_g * \nu(\Omega(\xi)). \end{aligned}$$

If $x \in \mathbb{T}$ then we can find $g \in \Gamma$ with $g o = x$, whence

$$q^{\mathfrak{h}(y_n)} G(x, y_n) \rightarrow c(\mu) \cdot \nu_x(\Omega(\xi)).$$

From this, the result follows. \square

The minimal P -harmonic functions. We can now determine the minimal harmonic functions. First of all we recall a well known fact that does not require any moment condition; see e.g. [27], Thm. 25.4.

(4.21) Proposition. *Let the function φ be defined as in (4.11). Then the minimal \tilde{P} -harmonic functions on \mathbb{Z} are precisely the functions $m \mapsto e^{cm}$, where $c \in \mathbb{R}$ is such that $\varphi(c) = 1$.*

Thus, if there is $c_0 \neq 0$ such that $\varphi(c_0) = 1$ then the positive \tilde{P} -harmonic functions on \mathbb{Z} are precisely the functions

$$m \mapsto t + (1 - t)e^{c_0 m}, \quad t \in [0, 1].$$

Otherwise, all positive \tilde{P} -harmonic functions are constant.

(4.22) Lemma. *If (y_n) is a sequence in \mathbb{T} tending to ω and such that $K(x, y_n) \rightarrow h(x)$ pointwise, then h depends only on $\mathfrak{h}(x)$, i.e., there is a \tilde{P} -superharmonic function f on \mathbb{Z} such that $h(x) = f(\mathfrak{h}(x))$ for all $x \in \mathbb{T}$.*

Proof. By the hypothesis, $u(o, y_n) \rightarrow \infty$. Thus, the proof is exactly as in (3.6), see Figure 2 and the subsequent lines. In general (unless P has finite range), the limit function is superharmonic, but not necessarily harmonic. \square

(4.23) Theorem. *Suppose that (i) $m(P) < \infty$ and $\alpha(P) > 0$, or that (ii) $m_{2+\varepsilon}(P) < \infty$ and $\alpha(P) = 0$, or that (iii) condition (4.14) holds.*

Then each function $K(\cdot, \xi)$, $\xi \in \partial^\mathbb{T}$ is minimal P -harmonic.*

In case (i), we have the following

- (a) *If the only solution to the equation $\varphi(c) = 1$ is $c = 0$, then the above are all minimal P -harmonic functions.*
- (b) *Otherwise, if $\varphi(c_0) = 1$ for some $c_0 \neq 0$ (whence $c_0 < 0$), the minimal P -harmonic functions are the above together with the function $h(x) = e^{c_0 \mathfrak{h}(x)}$.*

In cases (ii) and (iii), the minimal P -harmonic functions are the above together with the constant function $h(\cdot) \equiv 1$.

Proof. We start with a similar argument as in the proof of Theorem 3.4.

First of all, note that there are positive harmonic functions that are not constant on horocycles, namely the functions $K(\cdot, \xi)$ with $\xi \in \partial^*\mathbb{T}$. Therefore, there must be at least one minimal harmonic functions h with the same property. Then there is a sequence (y_n) in \mathbb{T} such that $K(\cdot, y_n) \rightarrow h$ pointwise. By compactness of $\hat{\mathbb{T}}$, we may assume that (y_n) converges in the end topology to a point of $\xi \in \partial\mathbb{T}$. It cannot be $\xi = \omega$, because in that case h would be constant on horocycles by Lemma 4.22. Thus, $\xi \in \partial^*\mathbb{T}$. Therefore, for this specific ξ , the function $K(\cdot, \xi)$ is minimal harmonic.

Now let $\eta \in \partial^*\mathbb{T}$ be arbitrary. Then there is $g \in \Gamma = \text{Aff}(\mathbb{T})$ such that $g\xi = \eta$. Note that we have the cocycle identity

$$(4.24) \quad K(x, g\xi) = K(g^{-1}x, \xi) / K(g^{-1}o, \xi).$$

Also, a function $h(x)$ is harmonic if and only if $h(gx)$ is harmonic. Using these observations, it is a straightforward exercise that $K(\cdot, \eta)$ is minimal as well. This proves the first part.

Suppose that h is a minimal harmonic function distinct from all $K(\cdot, \xi)$, $\xi \in \partial^*\mathbb{T}$. Then we must have, using the same argument as at the beginning of the proof, that $h = \lim K(\cdot, y_n)$ for a sequence (y_n) that tends to ω in the end compactification. Therefore h is constant on horocycles, that is, we can write $h(x) = f(\mathfrak{h}(x))$, where f is a \tilde{P} -harmonic function on \mathbb{Z} . Minimality of h with respect to P implies \tilde{P} -minimality of f . By Lemma 4.21, $h(x) = e^{c\mathfrak{h}(x)}$, where c satisfies $\varphi(c) = 1$.

Now suppose we are in case (i). Then

$$\int K(\cdot, \xi) d\nu_o(\xi) = 1,$$

whence the constant harmonic function 1 is not minimal harmonic. Thus, if $c = 0$ is the only solution of $\varphi(c) = 1$, then h as above cannot exist, and statement (a) holds.

Otherwise, we have to verify that $h(x) = e^{c_0\mathfrak{h}(x)}$ is indeed minimal. As stated, we must have $c_0 < 0$, since the (at least one-sided) derivative of φ at 0 is positive, and φ is convex.

Suppose that h is not minimal. Then the minimal harmonic functions are precisely the $K(\cdot, \xi)$, $\xi \in \partial^*\mathbb{T}$. Thus, there is a probability measure ν^h on $\partial^*\mathbb{T}$ such that

$$h(x) = \int_{\partial^*\mathbb{T}} K(x, \cdot) d\nu^h.$$

A straightforward computation based on Lemma 4.18 shows that for $r > k \geq 0$,

$$\sum_{x \in T_{k,r}} K(x, \xi) = q^r \frac{a_0 + \cdots + a_{r-1} + \kappa_k a_r}{a_k} \quad \text{for all } \xi \in \Omega_k,$$

where $\kappa_k = 1$ if $k \geq 1$ and $\kappa_0 = \frac{q-2}{q-1}$. Therefore

$$\begin{aligned} \left(\frac{q-1}{q}\right)^{\text{sign}(k)} q^r e^{c_0(r-k)} &= \sum_{x \in T_{k,r}} f(x) \geq \int_{\Omega_k} \left(\sum_{x \in T_{k,r}} K(x, \cdot) \right) d\nu^h \\ &\geq q^r \frac{a_0 + \cdots + a_{r-1} + \kappa_k a_r}{a_k} \nu^h(\Omega_k). \end{aligned}$$

We see that

$$a_k e^{c_0(r-k)} \geq (a_0 + \cdots + a_{r-1}) \nu^h(\Omega_k)$$

for all $r > k$. Letting $r \rightarrow \infty$, the left hand side tends to 0, while the right hand side tends to $\nu^h(\Omega_k)$. Thus, the measure ν^h vanishes everywhere, a contradiction.

In case (ii), the only solution of $\varphi(c) = 1$ is $c = 0$. The associated harmonic function is $h(\cdot) \equiv 1$. It is known from [7] that in the case when $\alpha(P) = 0$, the Poisson boundary is trivial, that is, all bounded harmonic functions are constant. This amounts to minimality of $h(\cdot) \equiv 1$. The same argument as used in case (i) shows that there can be no further minimal P -harmonic functions.

Case (iii) is immediate by applying case (i.b) to P^\sharp , since a function h is minimal harmonic for P if and only if $h^\sharp(x) = e^{-c_0\mathfrak{h}(x)}h(x)$ is minimal harmonic for P^\sharp . Here, $c_0 > 0$ is the constant of (4.14). \square

5. CONCLUSION

We now return to the “lamplighter setting”, where P is a semi-isotropic, irreducible transition matrix on $\text{DL}(q, r)$. We write P_1 and P_2 for the projections of P onto $\mathbb{T}^1 = \mathbb{T}_q$ and $\mathbb{T}^2 = \mathbb{T}_r$, respectively. We can apply all results of the preceding §4 to each P_i . If $\tilde{\mu}_i$ denotes the measure on \mathbb{Z} that describes the projection \tilde{P}_i of P_i , i.e., $\tilde{p}_i(k, l) = \tilde{\mu}_i(l - k)$, then $\tilde{\mu}_2(k) = \tilde{\mu}_1(-k)$, whence $\varphi_2(c) = \varphi_1(-c)$ for the associated functions according to (4.11). We shall stick to $\varphi = \varphi_1$, which in terms of P on DL is given by

$$\varphi(c) = \sum_{x_1 x_2 \in \text{DL}} e^{c\mathfrak{h}(x_1)} p(o_1 o_2, x_1 x_2),$$

and

$$\alpha(P) = \alpha(P_1) = \sum_{x_1 x_2 \in \text{DL}} \mathfrak{h}(x_1) p(o_1 o_2, x_1 x_2),$$

if the latter series converges absolutely (so that $\alpha(P_2) = -\alpha(P_1)$). The moments $m_t(P)$ are defined as in (4.7). In addition, we also introduce the exponential moment

$$m^{(c)}(P) = \sum_{x_1 x_2 \in \text{DL}} \left(d(o_1, x_1) e^{c_+ \mathfrak{h}(x_1)} + d(o_2, x_2) e^{c_- \mathfrak{h}(x_2)} \right) p(o_1 o_2, x_1 x_2),$$

where $c_+ = \max\{c, 0\}$ and $c_- = \min\{c, 0\}$. The purpose of this condition is the following. Suppose that there is $c_0 \neq 0$ such that $\varphi(c_0) = 1$ and $m^{(c_0)}(P) < \infty$. If $c_0 > 0$ then P_1 satisfies (4.14) on \mathbb{T}^1 , and Theorem 4.23 applies to P_1 . Also, $m_1(P_2) < \infty$ in that case, whence $\alpha(P_2)$ exists, and it must be $\alpha(P_2) > 0$, since $\alpha(P_1) = -\alpha(P_2)$ cannot be non-negative. Therefore, Theorem 4.23 also applies to P_2 . If $c_0 < 0$, the situation is analogous, with the roles of P_1 and P_2 exchanged.

In each case, we write $K_i(x_i, \xi_i)$ for the respective kernels on \mathbb{T}^i according to Proposition 4.17 and Lemma 4.18.

(5.1) Theorem. *Let P be an irreducible, semi-isotropic transition matrix on $\text{DL}(q, r)$, and P_i ($i = 1, 2$) its projections onto the trees $\mathbb{T}^1 = \mathbb{T}_q$ and $\mathbb{T}^2 = \mathbb{T}_r$, respectively. Suppose that (I) $m_{2+\varepsilon}(P) < \infty$ and $\alpha(P) = 0$, or that (II) there is $c_0 \neq 0$ such that $\varphi(c_0) = 1$ and $m^{(c_0)}(P) < \infty$.*

Then each of the functions $x_1 x_2 \mapsto K_i(x_i, \xi_i)$, where $\xi_i \in \partial^ \mathbb{T}^i$ ($i = 1, 2$) is a minimal P -harmonic function on $\text{DL}(q, r)$.*

In case (I), the minimal harmonic functions are the above together with the constant function $h(\cdot) \equiv 1$. In case (II), the above are all minimal harmonic functions.

Proof. Combining Theorem 3.4(a) with Theorem 4.23, we conclude that each minimal P -harmonic function must be of the form $x_1 x_2 \mapsto K_i(x_i, \xi_i)$ with $\xi_i \in \partial^* \mathbb{T}^i$ ($i = 1, 2$), or $x_1 x_2 \mapsto e^{c\mathfrak{h}(x_1)}$ with $\varphi(c) = 1$.

Since there are P -harmonic functions that depend only on x_i , but are not of the form $x_1 x_2 \mapsto f(\mathfrak{h}(x_i))$, there also must be a minimal harmonic function with these properties. By the above, it must be a kernel $x_1 x_2 \mapsto K_i(x_i, \xi_i)$. Hence, there is at least one $\xi_i \in \partial^* \mathbb{T}^i$ such that $x_1 x_2 \mapsto K_i(x_i, \xi_i)$ is minimal P -harmonic. Using the same cocycle argument as

below (4.24) in the proof of Theorem 4.23, we obtain that *all* $\xi_i \in \partial^* T^i$ ($i = 1, 2$) give rise to a minimal P -harmonic function on DL.

In case (I), by Theorems 3.4 and 4.23 the only other candidate for being a minimal P -harmonic function is the constant function $h(\cdot) \equiv 1$. The latter is indeed minimal: by Theorem 3.4, every positive bounded P -harmonic function is of the form $h(x_1 x_2) = h_1(x_1) + h_2(x_2)$, where each h_i must be bounded P_i -harmonic, whence constant by Theorem 4.23, as $\alpha(P_i) = 0$. Now recall that the constant function 1 is minimal if and only if all bounded harmonic functions are constant.

In case (II), the only candidates besides the kernels $K_i(\cdot, \xi_i)$ for being minimal P -harmonic are the functions $x_1 x_1 \mapsto e^{c\mathfrak{h}(x_1)}$ with $c = 0$ and $c = c_0$. Regarding $c = 0$, we know that the constant function $h(\cdot) \equiv 1$ is not minimal, since it is not minimal for P_i on \mathbb{T}^i , where i is the index for which $\alpha(P_i) > 0$. Analogously, if we define P^\sharp by $p^\sharp(x_1 x_2, y_1 y_2) = p(x_1 x_2, y_1 y_2) e^{c_0(\mathfrak{h}(y_1) - \mathfrak{h}(x_1))}$, then $h(\cdot) \equiv 1$ is not minimal for P^\sharp , whence $x_1 x_2 \mapsto e^{c_0 \mathfrak{h}(x_1)}$ is not minimal for P . \square

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